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Nonlinear liquidity-growth dynamics with corridor-stability

Willi Semmler

New School for Social Research, New York, NY 10003, USA

Malte Sieveking

University of Frankfurt, 6000 Frankfurt, Germany

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The paper presents a dynamic model of the financial/real interaction. In particular, it shows that (i) liquidity, when facilitated through credit, can operate procyclically, (ii) credit may add to the asymmetry of business cycles and (iii) endogenous propagation mechanisms in monetary economies are shock dependent. Using a variant of Foley's growth cycle model, we demonstrate that the portrayal of financial/read forces exhibits corridor-stability. In this case, small shocks have no lasting effects, but large enough shocks can lead to persistent cycles or unstable non-periodic fluctuations. The Hopf-birfurcation theorem is rendered inapplicable due to the fact that the trajectories are stable in the vicinity of equilibrium. A global characterization of the dynamics is required instead.

> 'As credit by growing makes itself grow, so when distrust has taken the place of confidence, failure and panic breed panic and failure' (Marshall 1879:99)

1. Introduction[1]

As the above citation indicates there is a long tradition taking the view that liquidity when facilitated through credit, may magnify cyclical expansions and contractions. Liquidity, of course, played a central role in Keynes' General Theory and in IS/LM variants of it. Recent advances in this vein have shown how various types of dynamic behavior arise from intrinsic monetary/real interactions. Various lines of work have embellished this basic theme and have emphasized several important implications of the financial role. First is that indeed liquidity can operate procyclically, amplifying business expansions or contractions. Second, the financial/real interaction is

Correspondence to: Willi Semmler, Department of Economics, Graduate Faculty, New School for Social Research, New York, 65 Fifth Ave., NY 10003, USA.

[1]We want to thank Richard Day for helpful suggestions concerning the exposition of the paper. We are also grateful for detailed comments by two referees of the journal.

asymmetric: the effect of financial variables are stronger in contractions than expansions. Third, endogenous propagation mechanisms in monetary economies may depend on the size of the shock.

The present contribution presents an analysis of these issues using a variant of Foley's (1986, 1987) growth cycle model. We obtain an explicit characterization of the different types of dynamics possible. In particular, we show that the model's portrayal of financial/real forces exhibits 'corridor-stability' in the sense of Leijonhuvud (1973). Small shocks to a system in a steady state have no lasting effects but large enough ones can lead to persistent cycles or unstable nonperiodic fluctuations.

After briefly discussing related literature in section 2, we lay out the main model in section 3. Like Foley we primarily focus on the behavior of the firm. Liquidity and productive assets of firms will be the two state variables of the system. A preliminary analysis of the dynamics is given in section 4; this is supported by simulation studies. Section 5 provides the mathematical analysis of the economic model by using perturbation analysis for nonlinear dynamics. Some concluding remarks are added in section 6.

2. Related literature

In the tradition of Keynesian theory, the monetary/real interaction has become central in IS/LM versions.[2] Usually, the asset market is represented by the money market.[3] There are interesting early nonlinear versions of an IS/LM macrodynamic [cf. Rose (1969), Torre (1977), Shinasi (1981)] which connect to recent work.

A novel contribution along this line is represented in the papers by Day and Shafer (1985) and Day and Lin (1989). As in the IS/LM version liquidity is provided from outside through exogenous money supply.[4] Money demand arises from transaction and liquidity motives. An infinitely fast adjustment process, through the adjustment of the interest rate, brings about a temporary equilibrium in the money market allowing the elimination of the interest rate as a variable. A boom then with a strain on liquidity chokes off the boom and the ease of liquidity in recessions allows for recoveries. An unstable accelerator effect destabilizes the system in the vicinity of the equilibrium. The monetary/real interaction generates intriguing periodic or nonperiodic fluctuations.

[2]There is already a tradition before Keynes that highlights the role of liquidity generated through credit for the business cycle; see for example, the theories of credit in Mill, Bagehot, and Marshall in the 19th century, von Hayek and Hawtrey since the 1920s, and Fisher in the 1930s. For an excellent survey on the earlier theorists, cf. Boyd and Blatt (1988).

[3]In the literature after Keynes the transaction and speculative demand for liquidity has become particularly central, cf. Modigliani (1944), Tobin (1958), Minsky (1975).

[4]A different variant is presented in Day (1989) where the money supply is not fixed but rather becomes a policy variable.

In Foley's various growth models (1986, 1987) – to be detailed below – money is presumed to grow at a fixed rate. In addition, commercial credit is introduced where firms are free to borrow and to lend. Banks provide loans and offer deposits so that the overall source of liquidity is commercial credit and deposits. Foley's model shows that an unstable accelerator, coupled with strong borrowing incentives by firms, produces instability in the vicinity of the equilibrium and that liquidity contains instability in the enterprise sector. The financial/real interaction – though in principle a three dimensional dynamic – results in periodic solutions studied through the Hopf-bifurcation theorem.

In models based on imperfect capital markets it is finance that plays a destabilizing role in macroeconomic activity, possibly amplifying business fluctuations. The reasons for this are first, an imperfect capital market – asymmetric information between lenders and borrowers and costly state verifications – drives a wedge between the internal and external cost of funds [Townsend (1979), Gale and Hellwig (1985), Bernanke and Gertler (1989)]. Second, default risk measured for example by balance sheet variables of firms gives rise to an increase in the cost of external finance which moves countercyclically accentuating the inverse relation between capital cost and investment [Bernanke and Gertler (1989, 1991), Greenwald and Stiglitz (1988), and Fazzari et al. (1988)].

In addition, the view that financial variables set in motion a stronger propagation mechanism of business activities is often paralleled by the hypothesis that the financial/real interaction also creates an asymmetry in the business cycle.[5] In particular, it is maintained that contractions are more strongly affected by financial variables than expansions.[6]

On the other hand, it is maintained that liquidity can serve as a buffer stock for flows smoothing production or consumption if the disturbances are not too large. Leijonhuvud (1973), for example, has argued that, in monetary economies, one should observe corridor-stability regarding macroaggregates. He shows that in an economy with buffer stocks small shocks to flows do not give rise to deviation amplifying fluctuations but large shocks may lead to a different regime of propagation mechanisms.

Finally, it is worth noting that there is strong empirical evidence supporting the view that liquidity covaries cyclically with investment and output.[7] A number of studies find procyclical credit flows, see, for example, Friedman

[5]Already in earlier nonlinear models it is demonstrated, for example, in Goodwin (1951) that contractions are asymmetric compared to expansions. There, an asymmetry arises due to a flexible accelerator; financial variables, however, are neglected in modeling cycles.

[6]This, for example, follows from the work of Bernanke (1981, 1983), and Mishkin (1978), who provide evidence for it for the Great Depression.

[7]The direction of causation remains controversial. It is still unresolved of whether money and credit lead output or output leads money and credit. For a recent evaluation of this matter, cf. Bernanke (1990).

(1983), and Blinder (1989). Blinder (1989), by decomposing credit market debt, shows that private credit market debt, in particular trade credit, moves strongly procyclically. The proposition that default risk and the (marginal) cost of external funds – as well as credit constraints – move countercyclically and are negatively correlated with investment and output is empirically demonstrated in Bernanke (1983), Gertler et al. (1991), and Franke and Semmler (1991).[8]

Given the theoretically and empirically well established role of financial variables in the business cycle we subsequently propose a growth cycle model which analytically studies the above issues.

3. The model

We commence with Folye's (1986, 1987) growth cycle version. The real side of the model is construed as follows. Firms through their capital outlay simultaneously determine their sales. Capital outlay, C, comprises the outlay for intermediate goods, wages (which are spent instantaneously) and an increase of capital stock, $\dot{K}$, which denotes an increase in the value of plant and equipment. Thus, investment is defined as part of the capital outlay. Prices are fixed. Wage income is instantly spent for consumption goods. Profit is solely saved by firms.

The financial side of the model can best be characterized by referring to the balance sheets of the economy [cf. Foley (1986)].

Balance Sheets

Assets		Liabilities
	Central Bank	
F_G		R
	Banks	
R		M
F_B		
	Firms	
M		D
F		
K		NW

where NW is the net worth of the sectors, F_G the central bank's holdings of loans, which is equal to the central banks reserve, R, and F_B is the banking sector's holding of loans to firms. Loans are also made among firms through commercial credit, which represent assets, F, for the lending firms. In order

[8]In those studies the cost of external funds is measured as spread between the 6 months commercial paper rate and the interest rate on treasury bonds.

to avoid problems of aggregate excess demand it is posited that money is directly transferred to firms. Thus we have $F_G + F_B + F = D$ or $M + F = D$.

The financial/real interaction can be portrayed by the ensuing three dimensional differential equation system [cf. Foley (1987)]. With profit $\Pi = qS$, q, the markup, S, Sales, the three ratios $m = M/K$, $f = F/K$, $r = \Pi/K$ entail the following growth rates from which a nonlinear differential equation system in m, f, r is derived:

$$\hat{m} = g - \hat{K}, \tag{1}$$

$$\hat{f} = (\dot{D}(m) - gM)/F - \hat{K}, \tag{2}$$

$$\hat{r} = a(r, m + f) - \hat{K}, \tag{3}$$

where g is the growth rate of money supply, $\hat{K}$ the growth rate of capital stock, $\dot{D}(m)$ is derived from $\dot{F} = \dot{D} - gM$, and $a(r, m+f) = \dot{C}/C = \dot{\Pi}/\Pi$ the growth rate of capital outlay (equal to the growth rate of profit flows). By assuming that liquidity and interest rate are inversely related the interest rate is eliminated as a variable in the model.

We propose the following modifications of the Foley model (1)–(3) which admit an explicit characterization of the possible dynamics. First, in the above eq. (1) we also allow for endogenously generated liquidity. We replace the constant g by the following function

$$g_l = g_l(g_m, r, \lambda),$$

where now g_m is a constant and $\lambda = L/C$. Accordingly in the above balance sheets of banks L is to be substituted for M. We emphasize the credit view of bank activities [cf. Bernanke (1990)]. Banks are free to issue debt (create deposits) in order to admit credit expansion in the enterprise sector.[9] The specification of the function $g_l(\cdot)$ is undertaken below.

Second, m from which $\hat{m}$ in eq. (1) is derived is the inverse of the velocity of money with respect to capital stock. We will, however, normalize through C – instead through K – since liquidity is typically not only used for investment in fixed Kapital, K, but also for working capital. Thus, λ expresses the inverse of the velocity of liquidity now not with respect to capital stock but with respect to capital outlay.[10]

Third, the behavioral function determining capital outlay $\dot{C}/C = a(r, m+f)$ is replaced by $\dot{C}/C = a(r, \lambda)$. It does not appear reasonable that f is an

[9] In addition the argument can be made, that beside bank loans and commercial papers, trade credit [Blinder (1989, ch. 5)] and unused credit lines [Huberman (1984)] are also important sources of liquidity for firms.

[10] In a later version, Foley has also adopted the above definition of the velocity [cf. Foley (1991)].

additional argument in the capital outlay function since for the enterprise sector as a whole the asset F is generated through the creation of debt D. We, therefore, include solely liquidity λ as an argument in the capital outlay function [for a similar view, cf. also Foley (1986)]. Correspondingly, for the growth rate of capital stock, $\hat{K}$, we also presume $\hat{K}=b(r,\lambda)$.

With those modifications the eq. (2) will not play a role any longer in the dynamics. We thus obtain a dynamic system in two variables only which reads as

$$\hat{\lambda}=g_l(g_m,r,\lambda)-\hat{C}(r,\lambda), \tag{5}$$

$$\hat{r}=\hat{C}(r,\lambda)-\hat{K}(r,\lambda). \tag{6}$$

This is the general form of our proposed dynamics.[11] With respect of $g_l(g_m,r,\lambda)$ of system (5), (6) two versions are explored. The first version we call Dynamic I. Here we define

$$g_l=g(g_m,r,\lambda) \text{ with } g_r,g_\lambda>0 \text{ everywhere.}$$

In this version we thus presume that the banks' willingness to hold the enterprise sector's debt depend positively on the rate of return and liquidity of firms. This expresses the fact that finance operates procyclically possibly magnifying expansions and contractions (as proposed by the above theories).

Subsequently, a second version is explored which we call Dynamic II. Here we define

$$g_l=g(g_m,r,\lambda)-h\ (r,\lambda),$$

where $g(g_m,r,\lambda)$ remains the same function as in Dynamic I but a function $h(r,\lambda)$ is added. This function is defined below. The term, $h(r,\lambda)$ represents a switch function activated only if r and λ fall below certain threshold values. This reflects the idea that finance adds an asymmetry to the financial/real interaction. We thus will add the term $h(r,\lambda)$ in a downswing. This expresses the view that liquidity will dissipate with the decline of cash flows and the deterioration of balance sheets of firms. Frequently, there are two arguments put forward that lead to dissipating liquidity. First, declining rates of return and deteriorating balance sheets give rise to an increase in the perceived riskiness of loans (default risk of borrowers) entailing a diminished willingness by lenders to buy the debt of firms. Second, with cash flows and liquidity dissipating agents will attempt to intertemporally transfer liquidity

[11]Note that in (6) we can also allow for constant proportion consumed out of profit flows. Let $c_\Pi qS$ be the consumed proportion of profits and $\Pi/K=qS/K$. We can substitute $S=C+c_\Pi qS$ and write $S(1-c_\Pi q)=C$, which gives, since c_Π and q are constants, $\hat{S}=\hat{C}$ and also $\hat{\Pi}=\widehat{(\mu S)}=\hat{S}$ therefore $\hat{\Pi}=\hat{S}=\hat{C}$. Observe also that eq. (6) represents a dynamic for the utilization of capacity for a growing economic system.

(preserving financial assets when bad times are expected which threaten the agents with possible insolvency and bankruptcy risk).[12]

4. The dynamics of the model

The following briefly discusses the above two types of dynamics. First Dynamic I will be elaborated were liquidity is provided in response to r and λ. The more complicated Dynamic II, resulting from the above function $h(\cdot)$, is studied thereafter.

4.1. Dynamic I

We specify our above defined function $g_l = g(g_m, r, \lambda)$ for the growth rates of liquidity as well as the functions $a(r,\lambda)$, $\hat{K}(r,\lambda)$ as linear functions. Also in the functions $a(\cdot)$ and $\hat{K}(\cdot)$ a constant will be included. The linear specification of our functions will give rise to nonlinear differential equations though of the simplest type.[13] We specify (5), (6) as

$$\hat{\lambda} = g_m + \theta_1 r + \theta_2 \lambda - (\beta_2 + \mu_1 r + \delta_1 \lambda), \tag{7}$$

$$\hat{r} = \beta_2 + \mu_1 r + \delta_1 \lambda - (\beta_1 + \gamma_1 r + \varepsilon \lambda). \tag{8}$$

In eq. (7) the first term on the right-hand side denotes our first version of $g_l(\cdot)$ and the term in brackets represents $a(\cdot)$ with β_2 the growth rate of the autonomous part of capital outlay and $\mu_1 r$, $\delta_1 \lambda$ the response of $a(\cdot)$ to the rate of return and the liquidity-capital outlay ratio respectively.

Eq. (7) can be simplified by using $\alpha = g_m - \beta_2$, $\beta = \theta_1 - \mu_1$ and $\varepsilon_1 = \theta_2 - \delta_1$. It seems to be empirically realistic to assume that β_2, θ_2 and θ_1 are small compared to g_m, δ_1 and μ_1, so that one expects negative signs for β and ε_1 and a positive sign for α. Then (7) can be rewritten as:

$$\hat{\lambda} = \alpha - \beta r - \varepsilon_1 \lambda. \tag{7'}$$

On the other hand, as shown, the growth rate of profit flows is determined by the growth rate of sales which is equivalent to the growth rate of capital outlays. Using the arguments for $a(\cdot)$ and $b(\cdot)$, we can write (8) as

$$\hat{r} = \beta_2 + \mu_1 r + \delta_1 \lambda - (\beta_1 + \gamma_1 r + \varepsilon \lambda). \tag{8'}$$

Again dropping unnecessary terms we denote $\gamma = \beta_2 - \beta_1$, $\varepsilon_2 = \mu_1 - \gamma_1$ and

[12]For details of those two arguments supporting the use of such a function $h(r,\lambda)$ in a macrodynamic model, cf. Bernanke (1981), and Bernanke and Gertler (1991). Empirical support of this view for the Great Depression is provided by Mishkin (1978) and Bernanke (1983).

[13]More general response functions could be employed but we want to explore the simplest case.

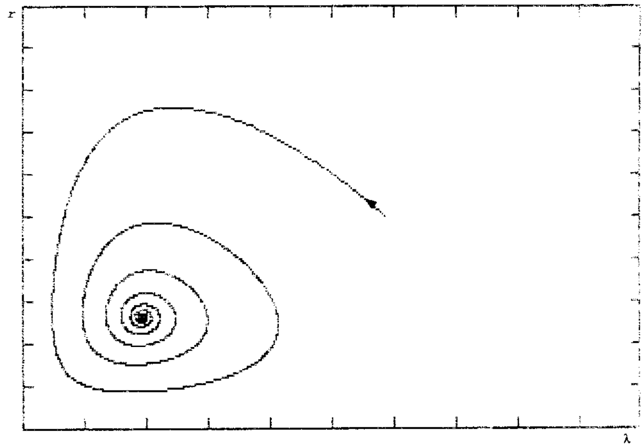

Fig. 1. Convergence of dynamic I.

$\delta = \delta_1 - \varepsilon$. Here, realism seems to suggest that $\beta_2 < \beta_1$, $\mu_1 < \gamma_1$ and $\delta_1 > \varepsilon$. By simplifying eqs. (7′) and (8′) as indicated above we can write our system of differential equations, called system (I), as

$$\hat{\lambda} = \alpha - \beta r - \varepsilon_1 \lambda,$$
$$\hat{r} = -\gamma + \delta\lambda - \varepsilon_2 r. \tag{9}$$

Equation system (9) is a nonlinear system of differential equations of Lotka-Volterra type – with ε_1 and ε_2 as perturbation terms. In system (9) no further perturbations appear yet. As will be demonstrated in section 5 the system (9) has three equilibria $(\lambda^* = 0, r^* = 0)$, $(\lambda^* = \alpha/\varepsilon_1, r^* = 0)$ and $(\lambda^* > 0, r^* > 0)$. The first two are saddle points and the last one is a globally attracting point. With the exception of those which start on one of the axes all of the trajectories converge to the unique attracting point $\lambda^* > 0$, $\gamma^* > 0$. The dynamic of system (9) is simulated by chosing economically realistic parameters.

For the simulation study the following parameters were used: $\alpha = 0.1$, $\gamma = 0.07$, $\varepsilon_1 = 0.045$, $\beta = 0.6$, $\delta = 0.7$, $\varepsilon_2 = 0.078$. The economically relevant equilibrium is $\lambda^* = 0.12$, $r^* = 0.15$.

Fig. 1 depicts the trajectories of system (9) where it shows that the trajectories, though they are oscillating, asymptotically approach the equilibrium $\lambda^* > 0$, $r^* > 0$.

4.2. Dynamic II

The second type of dynamic where the function $h(\cdot)$ is included is to be

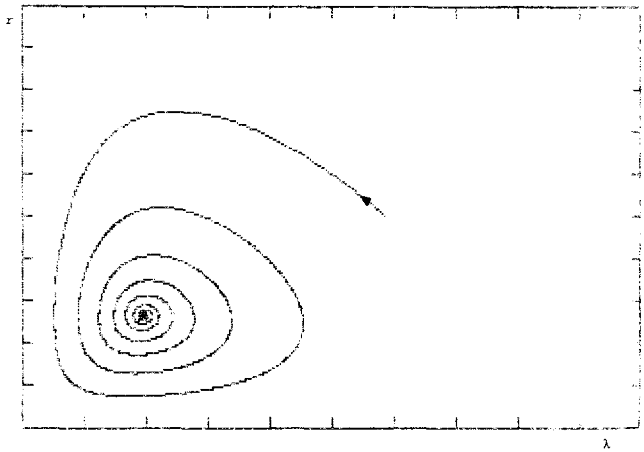

Fig. 2. Convergence case for dynamic II ($v = 0.2$).

elaborated. As aforementioned the function $h(\cdot)$ represents the idea that in business contractions lenders willingness to provide credit may depend strongly on the state of firms. In addition, agents faced with bankruptcy risk may tend to be reluctant to use liquidity for current spending (but tend to preserve financial assets for bad times). The dissipation of liquidity,[14] however, will entail a decline in capital outlay and investment of firms setting in motion a complicated dynamic.

Concerning the function $h(\cdot)$ we presume that if the rate of return falls below a certain rate of return ϕ ($r < \phi$), with $\phi < r^*$) or/and simultaneously liquidity drops below a certain ratio μ ($\lambda < \mu$) with $\mu < \lambda^*$ liquidity is dissipating, correspondingly affecting capital outlay and investment of firms. We then replace (9) through the following system of differential equations

$$\begin{aligned} \hat{\lambda} &= \alpha - \beta r - {}_1\lambda - h(\lambda, r), \\ \hat{r} &= -\gamma + \delta\lambda - \varepsilon_2 r. \end{aligned} \tag{10}$$

Formally, the term h in (10), is a control term in our dynamic system, representing the response of banks and firms to a decrease of the liquidity ratio below μ and the rate of return below ϕ. We shall assume that h in (10) is a smooth function satisfying

(i) $0 \leqq h(\lambda, r)$ $(\lambda \geqq 0, r \geqq 0)$,
(ii) $0 = h(\lambda, r)$ $(r \geqq 0, \lambda \geqq \gamma/\delta)$,

[14] One may also argue that a symmetric effect might occur in expansions. Since booms usually are resource constrained we want to neglect this slight complication.

(iii) $\alpha - h(\lambda, 0) > 0$ $(\lambda \geqq 0)$,
(iv) $h \neq 0$.

For the purpose of our computer simulation study we choose $h(\lambda, r) = v[\max(\phi - r, 0) \max(\mu - \lambda, 0)]^{1/2}$. The nonlinear differential eqs. (10) we call system (II).

The proposition that the system (I), represented by (9), is stable in the neighborhood of the equilibrium still holds for system (II), since the Jacobian for (II) is the same as for (I) at λ^*, r^*. Whereas the term h, pushes the trajectories toward the axes as soon as λ and r decline below μ and ϕ, the terms ε_1, and ε_2 generate attracting forces, keeping the trajectories in a compact set. The exact analytical study of the impact of the perturbation terms on the Lotka-Volterra dynamics is given in section 5.2 and 5.3. Here, it may suffice to illustrate the possible scenarios by again referring to a simulation study.

We can distinguish three scenarios. For a small reaction coefficient v the trajectories still converge toward the equilibrium values of λ, and r for any initial condition – similar to the trajectories of system (I). This case is analytically studied in section 5, remark 3. The simulation results are depicted in fig. 2.

For a greater reaction coefficient v the system (II) still converges toward the equilibrium for small shocks. For stronger shocks, i.e., for farther departure from the equilibrium values of λ, and r, however, system (II) becomes unstable until it finally approaches a limit cycle. On the other hand, for initial conditions, farthest away from the equilibrium, the limit cycle is approached from the outside. The existence of a limit cycle outside an asymptotically stable region is studied in the Theorem in section V and the simulation results are shown in fig. 3.

As also shown in fig. 3 for trajectories starting close to the equilibrium the system is stable. Only a stronger shock, i.e. initial conditions far enough from the equilibrium will generate limit cycles. Thus, the system exhibits corridor-stability.

In the last case depicted here, fig. 4, we have allowed the reaction parameter v to become even larger. The immediate effect is that the trajectory approaches zero. This problem is studied in remark 1 in section 5.

More generally, it can be shown that the system (II) has at least two limit cycles which can be revealed when time is reversed (cf. section 5).

5. Analytical treatment of systems (I) and (II)

The analysis of the equilibria and dynamics of our proposed systems (I) and (II) is undertaken as follows. Starting with the original Lotka-Volterra

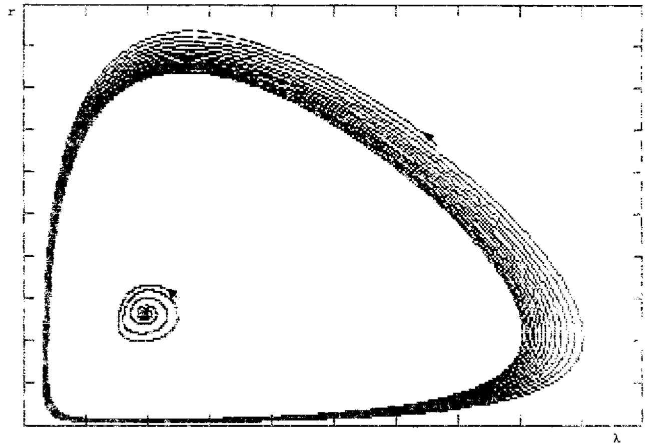

Fig. 3. Limit cycle of dynamic II ($v=0.6$).

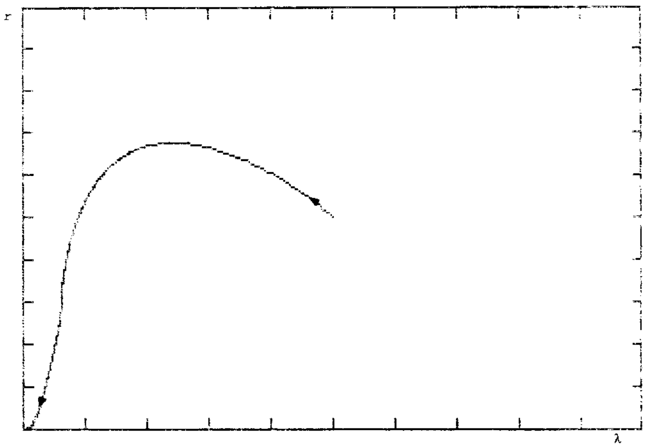

Fig. 4. Totally unstable dynamic II ($v=3$).

system the perturbations representing the attracting and the repelling forces will successively be introduced and the resulting dynamics studied.

5.1. *The Lotka-Volterra system*

The original Lotka-Volterra system is given by

$$\dot{\lambda}=\lambda(\alpha-\beta r)$$
$$\dot{r}=r(-\gamma+\delta\lambda) \text{ for } \lambda \geqq 0,\ r \geqq 0. \tag{11}$$

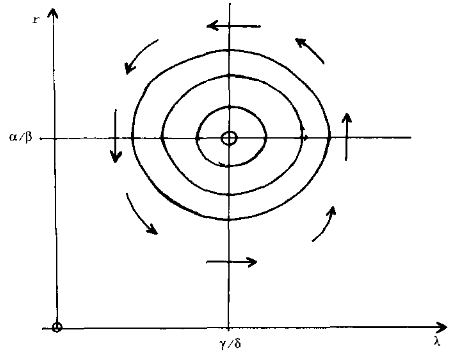

Fig. 5. Closed orbits of (11).

This system is most simply analyzed with the aid of the function $H(\lambda, r) = \alpha \log r + \gamma \log \lambda - \beta r - \delta \lambda$ $(\lambda > 0, r > 0)$ which is easily shown to be constant along trajectories of (11) with positive coordinates. These trajectories therefore coincide with the sets

$$H^{-1}(y) = \{(\lambda, r) \in (0, \infty)^2 \mid H(\lambda, r) = y\}.$$

As a consequence all of the orbits of (11) are closed orbits around $(\gamma/\delta, \alpha/\beta)$ with the exception of the following three: $(0,0)$ $\{0\}x[0, \infty)$ and $[0, \infty)x\{0\}$.

5.2. *A first perturbation*

Here we add a vector field to the system (11) which forces the trajectories to spiral outward. The analysis is greatly facilitated by the fact that the perturbation is confined to the region $\lambda \leqq \gamma/\delta$.

The perturbed system is given by

$$\dot{\lambda} = \lambda(\alpha - \beta r - h(\lambda, r)),$$
$$\dot{r} = r(-\gamma + \delta \lambda), \tag{12}$$

where h is a smooth function satisfying the conditions (i)–(iv) above. Let $a = (\gamma/\delta, a_2)$ be a point on the line $\lambda = \gamma/\delta$ having $a_2 > \alpha/\beta$ and let $(\lambda(t), r(t)) = x(t)$ be the solution of (12) starting at $x(0) = a$.

Claim 1. There is a first time $t > 0$ when $x(t)$ meets the segment $\{\gamma/\delta\}$ $x[\alpha/\beta, \infty)$ again and this happens at some $b = (\gamma/\delta, b_2)$ with $b_2(a_2) \geqq a_2$.

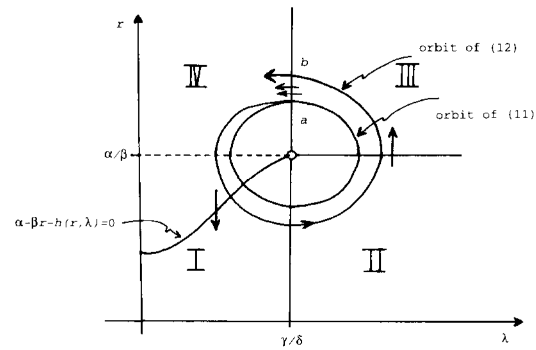

Fig. 6. Orbits of (12) spiraling outward.

Proof. It is easily seen that trajectories of the system (12) cannot enter the closed orbits of (11) or put equivalently that the function H increases along the trajectories of (12). Since both systems coincide for $\lambda > \gamma/\delta$ it suffices to show that $\lambda(t) = \gamma/\delta$ for at least one $t > 0$.

If this was not the case then $\lambda(t) < \gamma/\delta$ for all $t > 0$ and $r(t)$ was strictly decreasing. Now consider the limit set

$$L_\omega(a) = \left\{ y \mid \exists t_n \in [0, \infty) \lim_n t_n = +\infty, \quad \lim x(t_n) = y \right\}.$$

If $y = (y_1, y_2) \in L_\omega(a)$ then $y_2 = \inf_{t \geqq 0} r(t)$ and therefore $y_1 \leqq \gamma/\delta$ implies $y_2 = 0$. Now $L_\omega(a)$ is positively invariant and the only positively invariant subset of $[0, \gamma/\delta] \times \{0\}$ is $(0,0)$. Therefore $L_\omega(a) = \{(0,0)\}$, that is $\lim x(t) = (0,0)$ which is impossible, since by (iii) λ is increasing near $(0,0)$.

Claim 2. $b_2(a_2) > a_2$, if a_2 is properly chosen.

Proof. By property (iv) of the function h there is a point $c = (c_1, c_2)$ such that $0 < c_1 < \gamma/\delta$ and $h(c) > 0$. There is a solution x to (11) such that $x(0) = (\gamma/\delta, a_2)$ for some $a_2 > \alpha/\beta$ and $c = x(t)$ for some $t > 0$. The corresponding solution y to (12) with $y(0) = x(0)$ may go the same way as x for a while but will part form x not later than at time t. Therefore when it reaches $(\gamma/\delta, b_2)$ for some $b_2 > \alpha/\beta$ according to claim 1 we have $b_2 > a_2$.

5.3. A second perturbation

It is fairly obvious by now that all of the orbits of the system (12) except those which stay at the coordinate axis may be spiraling outward without

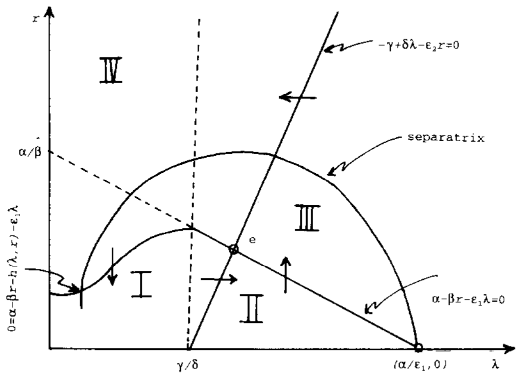

Fig. 7. Isoclines and equilibria of (13).

converging to a limit cycle. We therefore introduce a second perturbation which contracts the orbits in such a way that all the trajectories stay bounded but some of the spirals are retained – at least if the parameters ε_1, $\varepsilon_2>0$ are chosen sufficiently small. The system reads

$$\begin{aligned}\dot{\lambda}&=\lambda(\alpha-\beta r-h(\lambda,r)-\varepsilon_1\lambda)=f_1(\lambda,r),\\ \dot{r}&=r(-\gamma+\delta\lambda-\varepsilon_2 r)=f_2(\lambda,r).\end{aligned}\tag{13}$$

Claim 3. All of the trajectories of (13) are bounded no matter how small $\varepsilon_1>0$ and $\varepsilon_2>0$ are chosen.

Proof. Obviously $f_1(\lambda,r)<-$ for $\lambda>\lambda_0=(\alpha+1)/\varepsilon_1$, $r\geqq 0$. Therefore any trajectory eventually enters the region $\lambda<\lambda_0$ and stays there forever. But if $0\leqq\lambda\leqq\lambda_0$ and $r>r_0=(\delta\lambda_0+1)/\varepsilon_2$ then $f_2(\lambda,r)\leqq -r_0$. Therefore every trajectory eventually enters the box $[0,\lambda_0]x[0,r_0]$ and stays there forever.

Obviously $(0,0)$ is an equilibrium of (13). The remaining ones are easily determined by $(\alpha/\varepsilon_1,0)$ and $e=e_\varepsilon$, the intersection of the two lines $-\gamma+\delta\lambda-\varepsilon_2 r=0$ resp. $\alpha-\beta r-\varepsilon_1\lambda=0$. The Jacobian of (13) in $(0,0)$ reads

$$\begin{bmatrix}\alpha-h(0,0) & 0\\ 0 & -\gamma\end{bmatrix}.$$

Therefore $(0,0)$ is a saddle point, the r-axis (λ-axis respectively) being the stable respective unstable manifold. The Jacobian at $(\alpha/\varepsilon_1,0)$ is

$$\begin{bmatrix}-\alpha & (-\alpha/\varepsilon_1)\beta\\ 0 & -\gamma+\delta\alpha/\varepsilon_1\end{bmatrix}.$$

Assuming $\gamma/\delta<\alpha/\varepsilon_1$ (like in the figure above) $(\alpha/\varepsilon_1,0)$ is a saddle point, with the λ-axis being the stable manifold. The unstable manifold is formed by a trajectory (separatrix) which emanates from $(\alpha/\varepsilon_1,0)$ into the $\lambda>0$, $r>0$ region. Finally the Jacobian of e is

$$\begin{bmatrix} -\varepsilon_1 e_1 & -\beta e_1 \\ \delta e_2 & -\varepsilon_2 e_2 \end{bmatrix}$$

and both of its eigenvalues are seen to have negative real parts. Therefore e is a sink that is asymptotically stable.

5.4. Limit sets of the complete dynamical system

First recall the definition of the ω-limit set $L_\omega(c)$ of a solution $x(t)$ of (13) starting from $x(0)=c$:

$$L_\omega(c)=\{y=(y_1,y_2)\in R^2 \mid \text{there is a sequence } (t_n)\subset[0,\infty)$$
$$\text{such that } \lim x(t_n)=y \text{ and } \lim t_n=+\infty\}$$

Claim 4. If $c_1>0$, $c_2>0$ then neither $(0,0)$ nor $(\alpha/\varepsilon_1,0)$ is contained in $L_\omega(c)$.

Proof. Suppose $(\alpha/\varepsilon_1,0)\in L_\omega(c)$ and $c_1>0$, $c_2>0$. Then if $x(t)$ is the solution of (13) starting from $c=(c_1,c_2)$ there is $x(t)$ arbitrarily close to $(\alpha/\varepsilon_1,0)$. Now in region II and III (see figure above) r is increasing. Therefore if we follow $x(s)$ backwards for $s<t$ then $r(s)$ decreases, showing that if $(\alpha/\varepsilon,0)$ $L_\omega(c)$, then $\delta/\gamma,0)\in L_\omega(c)$. Since ω limit sets are positively and negatively invariant this implies that also $(0,0)\in L_\omega(c)$. It will therefore suffice to derive a contradiction from $(0,0)\in L_\omega(c)$.

Now if $x(t)$ is sufficiently close to $(0,0)$ it stays in region I where λ is increasing and r is decreasing. Therefore if $(0,0)$ $L_\omega(c)$ then necessarily $(0,y_2)\in L_\omega(c)$ for some $y_2>0$. Again since $L_\omega(c)$ is negatively invariant the whole segment $\{0\}x[0,+\infty)$ must belong to $L_\omega(c)$. But $x([0,+\infty))$ is bounded and therefore $L_\omega(c)$ too. We thus arrive at a contradiction which proves our claim.

Theorem. If $\varepsilon_1>0$ and $\varepsilon_2>0$ are sufficiently small then the system (13) *possesses at least two limit cycles. One of the limit cycles is obtained as the ω-limit set of the separatrix emanating from the equilibrium $(\alpha/\varepsilon_1,0)$. This limit cycle however contains a second one (which is unstable from the inside).*

Proof. Choose $a=(\gamma/\delta,a_2)$ according to claim 2 and consider the set Γ which is bounded by the curve $x([0,t])$ and $\{\delta/\gamma\}x[a_2,b_2]$. Here $x(s)$ is the solution of (12) which starts at a and meets $\{\delta/\gamma\}x[\alpha/\beta,\infty)$ for the second time at t in $b=(b_1, b_2)$. Since the vector field of (12) points strictly to the left on $\{\gamma/\delta\}x[\alpha/\beta, \infty]$ the set Γ may not be entered by any of the trajectories of (12). Now since solutions of (13) depend continuously on $\varepsilon=(\varepsilon_1, \varepsilon_2)$ the same is true for (13) if ε is sufficiently small: there is a set Γ_ε bounded by $x_\varepsilon[0, t_\varepsilon]$ and $\{\gamma/\delta\}x[a_2, b_{\varepsilon,2}]$ where x_ε is the solution of (13) which starts at a and meets $\{\gamma/\delta\}x[\alpha/\beta, \infty)$ a second time t_ε at $b_\varepsilon=(b_{\varepsilon,1}b_{\varepsilon,2})$. If $\varepsilon>0$ is sufficiently small none of the trajectories of (13) ever enters Γ_ε from outside and in addition Γ_ε contains the equilibrium $e=e_\varepsilon$ (cf. sect. V.3) in its interior.

Now consider the limit set L of the separatrix emanating from $(\alpha/\varepsilon, 0)$. Since the separatrix cannot enter Γ_ε, L does not contain the equilibrium $e=e_\varepsilon$. According to claim 4 L contains neither $(0,0)$ nor $(\alpha/\varepsilon_1, 0)$. Since L is compact by claim 3 and does not contain an equilibrium it is a limit cycle by Poincaré–Bendixson's theorem. Now consider a trajectory for the reversed system of (13)

$$\dot{\lambda} = -\lambda(\alpha - \beta r - h(\lambda, r) - \varepsilon_1 \lambda),$$
$$\dot{r} = -r(-\gamma + \delta\lambda - \varepsilon_2 r). \tag{13*}$$

If the trajectory starts within Γ_ε it may not leave it and therefore has a compact nonvoid limit set L^-. Since L^- is contained in Γ_ε it does not contain any of the equilibria $(0,0)$ or $(\alpha/\varepsilon, 0)$ but neither e_ε since – by section V.3 – e_ε is a repellor for (13*). Therefore – again by Poincaré–Bendixson's theorem – we conclude that L^- is a limit cycle, contained in the interior of the first one. Q.E.D.

Remark 1. Suppose h is increased in such a way that condition (iii) is violated and we have in fact

$$\alpha - h(0,0) < 0.$$

Then $(0,0)$ is asymptotically stable and for a nonvoid open subset of starting points the system collapses to $(0,0)$. Let us assume that the set $\alpha-\beta r-\varepsilon_1\lambda-h(\lambda, r)=0$ is a curve which cuts the λ axis exactly once between $\lambda=0$ and $\lambda=\gamma/\delta$. The intersection point $\tilde{e}=(\tilde{e}_1, \tilde{e}_2)$ is an equilibrium which in fact will be a saddle point. Consequently there is a separatrix s_1 emanating from $\tilde{e}$ and we may distinguish two cases with respect to the position of s_1 relative to the separatrix s_2 emanating from $(\alpha/\varepsilon_1 . 0)$.

Case 1. s_2 lies above s_1 (see fig. 8). In this case the limit set of s_1 for the time reversed system is a limit cycle because for the reversed system e is a repellor (source).

Case 2. s_2 lies below s_1 (see fig. 9). In any case a trajectory will tend to $(0,0)$ if it cuts the line $-\gamma+\delta\lambda-\varepsilon_2 r=0$ above s_1. If however s_2 tends to e – which is at least conceivable – then in case 2 there are only two possibilities:

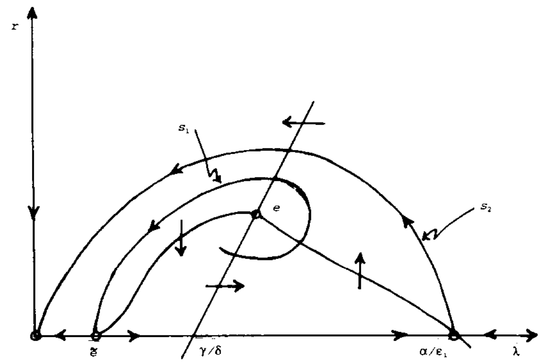

Fig. 8. s_2 lying above s_1.

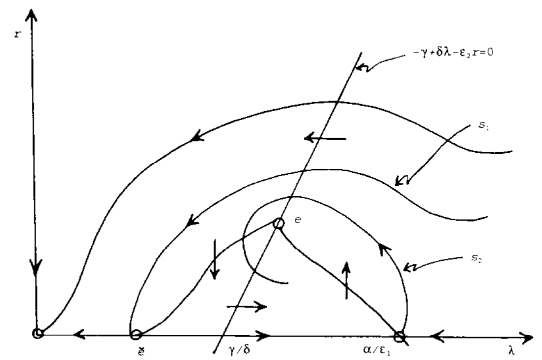

Fig. 9. s_2 below s_1.

either a trajectory converges to $(0,0)$ or to e, exception made by s_1, and those trajectories which lie on the λ axis to the right of $\tilde{e}$.

Remark 2. If $\alpha/\varepsilon_1<\gamma/\delta$ then every solution $x(t)$ of (13) converges to the λ-axis as t tends to infinity. This is because for $\lambda>\gamma/\delta$ $\dot{\lambda}$ is negative and for $\lambda<\gamma/\delta$ $\dot{r}$ is negative (see fig. 10).

Remark 3. We shall show that $e=e_\varepsilon$ is a global attractor for (13) if $\varepsilon=(\varepsilon_1,\varepsilon_2)$ is kept fixed and h is made so small that

$$h(\lambda,\lambda)<\varepsilon_1(e_1-\lambda) \text{ for } \lambda<e_1.$$

To do so we first consider the system

$$\begin{aligned}\dot{\lambda}&=\lambda(\alpha-\beta r-\varepsilon_1\lambda),\\ \dot{r}&=r(-\gamma+\delta\lambda-\varepsilon_2 r).\end{aligned} \tag{14}$$

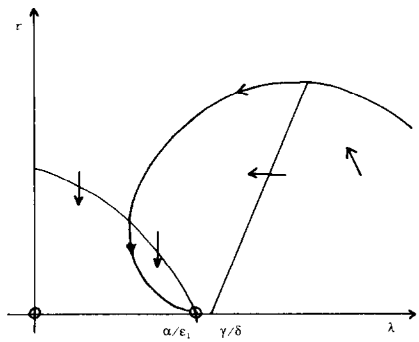

Fig. 10. Vanishing of r for $\alpha/\varepsilon_1 < \gamma/\delta$.

Using the fact that $e = e_\varepsilon = (e_1, e_2)$ is an equilibrium for (14) this may be rewritten as

$$\begin{aligned} \delta\lambda^{-1}\dot{\lambda}(e_1-\lambda) &= \beta\delta(e_2-r)(e_1-\lambda)+\delta\varepsilon_1(e_1-\lambda)^2 \\ \beta r^{-1}\dot{r}(e_2-r) &= -\beta\delta(e_2-r)(e_1-\lambda)+\beta\varepsilon_2(e_2-r)^2. \end{aligned} \tag{15}$$

Taking the sum we find

$$\delta\lambda^{-1}\dot{\lambda}(e_1-\lambda)+\beta r^{-1}\dot{r}(e_2-r)=\delta\varepsilon_1(e_1-\lambda)^2+\beta\varepsilon_2(e_2-r)^2$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}H(x(t))=\frac{\partial H}{\partial\lambda}\dot{\lambda}+\frac{\partial H}{\partial r}\dot{r}=-\delta\varepsilon_1(e_1-\lambda)^2-\beta\varepsilon_2(e_2-r)^2,$$

where $H(\lambda, r) = \delta(\lambda - e_1 \log \lambda) + \beta(r - e_2 \log r)$.

Therefore H is a global Liapunov function for (14). Let us investigate if H is also a Liapunov function for (13).

Let $y(t)$ be a solution of (13). Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}H(y(t)) &= \frac{\partial H}{\partial y_1}\dot{y}_1+\frac{\partial H}{\partial y_2}\dot{y}_2=\frac{\partial H}{\partial y_1}(\alpha-\beta y_2-\varepsilon_1 y_1) \\ &\quad+\frac{\partial H}{\partial y_2}y_2(-\gamma+\delta y_1-\varepsilon_2 y_2)-\frac{\partial H}{\partial y_1}y_1 h(y_1,y_2) \\ &= -\delta\varepsilon_1(e_1-y_1)^2-\beta\varepsilon_2(e_2-y_2)^2-\delta(y_1-e_1)h(y_1,y_2), \end{aligned}$$

and this is negative for $y \neq e$ in case $h(y_1, y_2) < \varepsilon_1(e_1 - y_1)$ for $0 \leqq y_1 < e_1$. In that case H is a Liapunov function for the system (13) too and hence e is a global attractor.

6. Some conclusions

As suggested in the paper, our proposed dynamics with the control term h satisfying properties (i)–(iv) above can be considered an enriched formalization of the role of liquidity in macrodynamics. A drain on financial liquidity of firms initiated by falling rates of return and dissipating cash balances, can give rise to scenarios that many economists have linked to the role of money and credit in macrodynamics. In fact, in our model different scenarios can arise according to different types of shocks and different values of the parameters of the system. If all other quantities are held fixed and $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are made small enough limit cycles will occur outside a stable vicinity of the equilibrium with positive coordinates. Technically, and contrary to other models, the equilibrium does not have to be unstable in order to generate a macroeconomic limit cycle. Such a model with corridor-stability, however, results in some technical difficulties to analyze the dynamics since the well-known Hopf-bifurcation theorem cannot be applied. An extension of the Poincaré–Bendixson theorem, developed in Sieveking (1988), was utilized instead.

By way of concluding we want to remark that the following problems may warrant a further study. First, one can turn the differential eqs. (10) into a problem of optimal control where h is the control variable and a suitable defined value of the firm is to be maximized by chosing h in the best possible way. In this context the question will then naturally arise whether the optimally controlled system exhibits a cyclic behavior again.[15] Second, it still remains a task to estimate the periods of the occurring limit cycles. In particular it may be interesting to compare the frequency of the undisturbed Lotka–Volterra system with the one of the disturbed system. Both problems are left open for future research.

[15]Limit cycles are also admitted in optimally controlled systems, cf. Semmler and Sieveking (1991).

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