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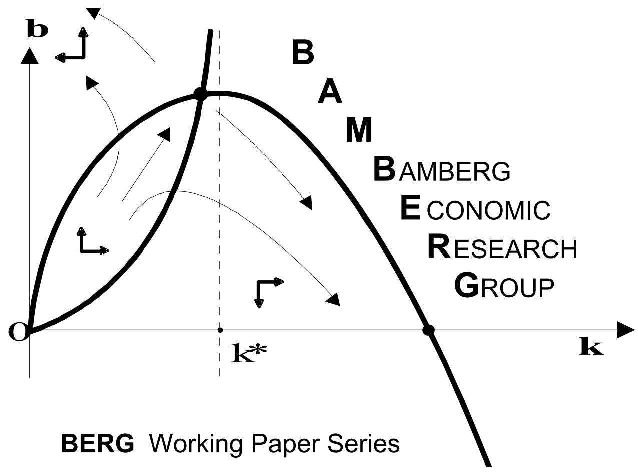

Bamberg Economic Research Group
Bamberg University
Feldkirchenstraße 21
D-96052 Bamberg
Telefax: (0951) 863 5547
Telephone: (0951) 863 2687
felix.stuebben@uni-bamberg.de
http://www.uni-bamberg.de/vwl/forschung/berg/

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Redaktion:
Dr. Felix Stübben*

* felix.stuebben@uni-bamberg.de

Coevolution of stock prices and their perceived fundamental value*

Sarah Mignot[a]
[a]Department of Economics, University of Bamberg, Germany

Abstract

We develop a simple nonlinear stock market model in which speculators switch between technical and fundamental trading rules depending on market conditions. Additionally, we assume that agents are unaware of the true current fundamental value and, thus, use a weighted average of the current price and the known long-run fundamental value as an estimate of the fundamental price. Using analytical and numerical methods, we demonstrate that an increase in the reaction parameter of technical traders may cause boom-bust dynamics. Moreover, we show that a heightened belief among agents that the fundamental value is more sensitive to deviations of the current price from its long-run fundamental value can cause the price to become trapped above or below this long-run value, oscillate within a higher price range, and prolong the duration of a bubble. In two model extensions, we assume that agents compute the current fundamental value based on the deviation between the average price and the known long-run fundamental value, using a moving average of the past k prices and an exponential moving average, respectively. These robustness checks show that, in these cases, price and perceived fundamental value fluctuate less statically around the long-run fundamental value.

Keywords
Bifurcation analysis; chartists and fundamentalists; boom-bust dynamics;

JEL classification
C62; D84; G10; G41

1 Introduction

Shiller (1981) argues that stock prices often exhibit excessive volatility, driven by factors far beyond what could be explained by changes in dividends. Shiller (2015) suggests that boom and bust dynamics in financial markets are driven by psychological factors such as anchors, overconfidence, and narratives.

The aim of our paper is threefold. The first aim is to assume a non-constant perceived fundamental value. To model agents' perception of the current fundamental value, we employ the anchoring and adjustment heuristic introduced by Tversky and Kahneman (1974) and expanded by Shiller (2015), which posits that agents use quantitative anchors as reference points for stock price levels. In fact, Tversky and Kahneman argue that agents form expectations based on an anchor and then make some adjustment from that anchor to obtain their estimate. In our paper, we assume that agents form their expectation

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about the current fundamental value by using the known long-run fundamental value as an anchor and, from that point, adjust their expectation with the current perceived distortion from the long-rung fundamental value. The second aim is to explore how this perceived fundamental value and the stock price coevolve. Finally, we examine how this perceived fundamental value influences the distortion in the stock market.

Our setup is based on the seminal works of Day and Huang (1990), Huang and Day (1993), and Chiarella (1992), incorporating three market participants, namely chartists, fundamentalists and a market maker. The price is driven by the demand of the chartists and fundamentalists, and the market maker who sets the price with respect to their excess demand. Chartists following the technical trading rule believe in the persistence of bull and bear markets. Fundamentalists, on the other hand, following the fundamental trading rule believe that the price will return to its fundamental value. Moreover, depending on the market conditions following De Grauwe et al. (1993) and He and Westerhoff (2005), we allow agents to switch between technical and fundamental trading rules. In particular, the market share of fundamentalists increases with the perceived mispricing. Finally, we assume that market participants do not know the true current fundamental value and therefore use a weighted average between the current price and the known long-run fundamental value as an indicator of the current fundamental price.

It turns out that our model is driven by a one-dimensional nonlinear map which possesses one fundamental steady-state and two non-fundamental steady-states. Our model shows that: (i) An increase in the reaction parameter of chartists can lead to a sequence of bifurcations which can cause the price to switch between bull and bear markets. (ii) If agents believe that the current fundamental value reacts more strongly to the deviation of the current price from its long-run fundamental value, then price dynamics may be trapped above or below the long-run fundamental value, price dynamics fluctuate in a higher price range and the duration of a bubble is longer.

In our second and third setup, we assume that agents compute the current fundamental value from the deviation between the average price and the known long-run fundamental value, using as a measure of the average price a moving average of the k past prices and an exponential moving average, respectively. These two cases can be regarded as a robustness check of our model. Moreover, the simulations reveal that price and perceived fundamental dynamics fluctuate less statically around the long-run fundamental value.

We proceed as follows. In Section 2, we provide a brief overview of different approaches to modelling non-constant fundamental value perception. In Section 3, we present the core of our model and provide analytical and numerical results. In Section 4, we consider the moving average case. In Section 5, we consider the exponential moving average case. In Section 6, we conclude our paper. Appendix A provides further analytical proofs related to the secondary bifurcations of our core model.

2 Literature review

In this section, we will briefly discuss some modelling of non-constant fundamental value perception.

One approach links fundamental value perception directly to the real economy. Studies by Westerhoff (2012), Lengnick and Wohltmann (2013), and Huang and Zhang (2017) propose that the fundamental value perceived by agents is proportional to the level of national income.

Another approach emphasizes the role of information in shaping perceptions. Diks and Dindo (2008) define the fundamental value as the ratio of expected future dividends to the difference between the interest rate and an estimated growth rate. In this model, informed agents have accurate knowledge of future dividends, while uninformed agents base their perceptions on current market prices and their own dividend estimates.

Psychological biases provide another perspective. De Grauwe and Kaltwasser (2012), Kaltwasser (2010), and Naimzada and Pireddu (2015) suggest optimistic and pessimistic fundamentalists who overestimate and underestimate the true fundamental value, respectively. In the same line, Gardini et al. (2022, 2024) propose a piecewise model in which the perceived fundamental value is influenced by current price trends. In this model, fundamentalists perceive a high fundamental value during positive trends and a low fundamental value during negative trends, while their perception remains unbiased during neutral trends.

Anchoring, where agents use psychological thresholds to estimate fundamental value, is another significant psychological bias. Huang et al. (2010) and Huang and Zheng (2012) assume that chartists' short-term estimate of the fundamental value is determined by dividing the price domain into n regimes defined by psychological thresholds, and setting the estimated fundamental value at the midpoint of the regime into which the current price falls. Similarly, Westerhoff (2003a) suggests that traders anchor their perceptions to the nearest round number, with the price domain divided into K round anchors. The midpoint of two consecutive round anchors defines the threshold, determining which anchor is relevant. Depending on the current exchange rate, the perceived fundamental value is then anchored to the nearest round number.

Finally, Westerhoff (2003b) explores three versions of perceived fundamental value. The first version assumes a normal distribution around the true fundamental value, similar to the methods used by De Grauwe and Grimaldi (2006) and Mignot and Westerhoff (2024). The second assumes that the perceived fundamental value is a weighted average of the past perceived fundamental value and the current perceived exchange rate, and a normally distributed shock. The third version incorporates a feedback learning mechanism, allowing agents to adjust their perceptions based on past errors.

All of these models illustrate the multifaceted nature of fundamental value perception, integrating real economic variables, information availability, psychological biases and anchoring within a framework of bounded rationality.

3 The general model

3.1 Model setup

In this section, we present our model setup, which adopts the market maker models of Day and Huang (1990), Huang and Day (1993), and Chiarella (1992). The starting point is that a market maker adjusts the price with respect to the excess demand of agents. We assume that the agents either opt for a technical trading rule or a fundamental trading rule, depending on the current market conditions.

The price formation is determined by a price adjustment function. Hence, the price P at time $t+1$ is given by

$$P_{t+1} = P_t + aED_t, \tag{1}$$

where parameter a is a positive price adjustment parameter and ED_t is the excess demand. The price increases (decreases) when there is an excess demand (excess selling).

We assume that N agents place orders by either opting for a technical trading rule, denoted by D_t^C, or for a fundamental trading rule, denoted by D_t^F. The excess demand is thus defined by

$$ED_t = N(N_t^C D_t^C + N_t^F D_t^F), \tag{2}$$

where N_t^C and N_t^F are the market shares of agents following the technical and fundamental trading rules, respectively. We normalise the mass of agents $N = 1$.

Agents following the technical trading rule, also called chartists, believe in bull and bear markets. When the price is below (above) the perceived fundamental value, they consider the market to be bearish (bullish) and submit selling (buying) orders. Orders placed by a single chartist are formalised by

$$D_t^C = b(P_t - F_t), \tag{3}$$

where b is a positive reaction parameter.

Agents following the fundamental trading rule, also called fundamentalists, assume that the price will return to its fundamental value. When the price is below (above) the perceived fundamental value, they submit buying (selling) orders. Orders placed by a single fundamentalist are formalised by

$$D_t^F = c(F_t - P_t), \tag{4}$$

where c is a positive reaction parameter.

The market share of chartists and fundamentalists is defined as

$$N_t^C = \frac{1}{1 + e + g(F_t - P_t)^2} \tag{5}$$

and

$$N_t^F = 1 - N_t^C, \tag{6}$$

respectively. When the price is close to the perceived fundamental value, a high proportion of agents perceive this as the start of a bubble, and the share of chartists is close to $\frac{1}{1+e}$. However, agents believe that bubbles will not persist but will burst at some time if the price is too far from its fundamental value. Therefore, when they perceive an increase in the mispricing, an increasing number of them switch to the fundamental trading rule. Parameter $e \geq 0$ determines the maximum share of chartists. If, for example, $e = 0.25$ and $F_t = P_t$, then $N_t^C = \frac{1}{1+e} = 0.8$. Parameter $g > 0$ is the switching parameter. The higher the value of parameter g, the faster the agents switch to the fundamental trading rule when they perceive an increase in the mispricing. The bell-shaped market share function is derived from De Grauwe et al. (1993), who applied it to a foreign exchange market. See He and Westerhoff (2005) and Dieci and Westerhoff (2010) for some other economic applications.

We assume that agents do not know the true current fundamental value due to a lack of perfect information. We further assume that they believe that the current fundamental value fluctuates around the long-run fundamental value F. This long-run fundamental value is known and serves as an anchor, i.e., as a starting point for their current perception of the fundamental value. Depending on the market conditions, they then adjust their current expectation of the current fundamental value upward or downward. If the current price is below (above) F, they believe that the current fundamental value is lower (higher) than F. We thus model the current perceived fundamental value as

$$F_t = F + d(P_t - F) = (1 - d)F + dP_t, \tag{7}$$

where $0 < d < 1$ indicates how strongly agents believe that the current fundamental value reacts to the deviation of the current price from F.

3.2 Analytical results

Combining (1) to (7) reveals that our model is driven by a one-dimensional nonlinear deterministic map, given by

$$P_{t+1} = P_t + a(1-d)(F-P_t)\frac{cg(F-P_t)^2(1-d)^2 + (ce-b)}{1+e+g(1-d)^2(F-P_t)^2}. \tag{8}$$

Setting $P_{t+1} = P_t = \overline{P}$, we obtain three steady-states: A fundamental steady-state given by

$$\overline{P_\star} = F \tag{9}$$

with $\overline{N_\star^C} = \frac{1}{1+e}$, and two non-fundamental steady states

$$\overline{P_\pm} = F \pm \frac{1}{1-d}\sqrt{\frac{b-ce}{cg}} \tag{10}$$

with $\overline{N_\pm^C} = \frac{c}{b+c}$. The fundamental steady state always exist whereas the two non-fundamental steady states only exist for $b \geq ce$.

The partial first-order derivative of map (8), evaluated at $\overline{P_\star}$, reads

$$\frac{dP_{t+1}}{dP_t}\left(\overline{P_\star}\right) = 1 + a(1-d)\frac{b-ce}{1+e}. \tag{11}$$

The model's fundamental steady state is locally stable if $|\frac{dP_{t+1}}{dP_t}\left(\overline{P_\star}\right)| < 1$, resulting in:

$$ce - \frac{2(1+e)}{a(1-d)} < b < ce. \tag{12}$$

The partial first-order derivative of map (8), evaluated at $\overline{P_\pm}$, reads

$$\frac{dP_{t+1}}{dP_t}\left(\overline{P_\pm}\right) = 1 - 2ac(1-d)\frac{b-ce}{b+c}. \tag{13}$$

The model's non-fundamental steady states are locally stable if $|\frac{dP_{t+1}}{dP_t}\left(\overline{P_\pm}\right)| < 1$, resulting in:

$$ce < b < ce + \frac{c(1+e)}{-1+ac(1-d)} \quad \text{for} \quad ac(1-d) > 1, \quad ce < b \quad \text{for} \quad ac(1-d) \leq 1. \tag{14}$$

We have thus proven the following proposition.

Proposition 1 ***(primary bifurcations)*** *Map (8) may possess up to three steady states: One fundamental steady state $\overline{P_\star} = F$ and two non-fundamental steady states $\overline{P_\pm} = F \pm \frac{1}{1-d}\sqrt{\frac{b-ce}{cg}}$. The fundamental steady state always exists. For $b^{PD1} < b < b^P$, $\overline{P_\star}$ is locally stable. A period-doubling bifurcation occurs at $b^{PD1} = ce - \frac{2(1+e)}{a(1-d)}$. At $b^P = ce$, a pitchfork bifurcation occurs and two non-fundamental steady states emerge and lie symmetrically around $\overline{P_\star}$. $\overline{P_\pm}$ are locally stable if either $b^P < b < b^{PD2}$ or $b^P < b$ and $ac(1-d) \leq 1$. A period-doubling bifurcation occurs at $b^{PD2} = ce + \frac{c(1+e)}{-1+ac(1-d)}$ if $ac(1-d) > 1$.*

Let us discuss the main economic implications of Proposition 1

- **Fundamental steady state**. The model's fundamental steady state implies that $\overline{N^C_\star} = \frac{1}{1+e}$, i.e., $\overline{N^C_\star}$ decreases in line with parameter e. The fundamental steady state stability condition can also be written as $\overline{N^C_\star}b - \overline{N^F_\star}c > -\frac{2}{a(1-d)}$ and $\overline{N^F_\star}c - \overline{N^C_\star}b > 0$. Violation of the first inequality results in a period-doubling bifurcation, which occurs when fundamental traders become too aggressive. However, this scenario requires extreme values for c. Violation of the second inequality results in a pitchfork bifurcation, which occurs when technical traders become too aggressive.

- **Non-fundamental steady states**. The non-fundamental steady states increasingly (decreasingly) deviate from F as parameter b and/or d (c, e and/or g) increase. The model's non-fundamental steady states imply that $\overline{N^C_\pm} = \frac{c}{b+c}$. Parameter c (b) increases (decreases) the share of chartists. The non-fundamental steady states' stability condition can also be written as $\overline{N^C_\pm}(b - ce) < \frac{1}{a(1-d)}$. The local stability of the non-fundamental steady states depends on all parameters except g. An increase in parameters a and b may compromise their local stability, while an increase in parameters e and d may be beneficial to their local stability.

The following proposition, proven in Appendix A, summarizes our results of map (8)'s secondary bifurcations.

Proposition 2 ***(secondary bifurcations)*** *Define $\delta = 1 - d$. For $b^P < b < b^H$, the dynamics are either trapped between the local maximum value and its image or between the local minimum value and its image. For $b^{PD2} < b < b^{PD4}$, two symmetric period-two cycles are locally stable. At $b^{PD4} = ce + \frac{(1+e)}{-1+ac\delta}\left(\frac{5+\sqrt{5}\sqrt{5+4ac\delta(ac\delta-2)}}{2a\delta} - c\right)$ the second period-doubling bifurcation occurs. A homoclinic bifurcation of the fundamental steady state occurs at $b^H = ce + \frac{(1+e)}{-1+ac\delta}\left(\frac{16+13ac\delta(-2+ac\delta)+2(4+3ac\delta(-2+ac\delta))^{\frac{3}{2}}}{a\delta(5+4ac\delta(-2+ac\delta))} - c\right)$. For $b^H < b < b^F$, the dynamics are bounded between the local maximum value and the local minimum value. A final bifurcation occurs at $b^F = ce + \frac{2c(1+e)}{-2+ac\delta}$ if $ac\delta > 2$. For $b > b^F$ the dynamics are divergent.*[1]

Let us discuss the implication of Proposition 2. For $b^P < b < b^H$, for an initial value lying in the basin of attraction $\mathcal{B}^+$, dynamics are attracted above F, while for an initial value lying in the basin of attraction $\mathcal{B}^-$, dynamics are attracted below F. These basins of attraction are the union of infinitely many intervals, having as limits F and the points of the unstable period-two cycle $\alpha_\pm = F \pm \frac{1}{1-d}\sqrt{\frac{2(1+e)+a(1-d)(b-ce)}{g(-2+ac(1-d))}}$ (this period two exists if $b > b^{PD1}$ and if $ac(1-d) > 2$).[2] The dynamics are either trapped in the bull market between the local maximum value and its image or in the bull market between the local minimum value and its image. At b^H, the two attractors and their basins of attraction merge. For $b^H < b < b^F$, we then observe dynamics in the bull and bear market, bounded between the local minimum value and the local maximum value. An increase in parameters e and d may cause the dynamics to be trapped in either the bull or the bear market, while an increase in parameters a and b might cause bull and bear market dynamics. The final bifurcation occurs at b^F, i.e., when the local maximum value equals α_+ and the local minimum value equals α_-. We have divergent dynamics for

[1] The final bifurcation is the last change in a system's dynamical behavior as a parameter is varied. After this point, the generic trajectory is divergent.

[2] $\mathcal{B}^+ := \mathcal{B}^+_0 \cup \mathcal{B}^+_{-1} \cup \mathcal{B}^+_{-2} \ldots$ $\mathcal{B}^+_0 =]F, \mathcal{O}_{-1}[$, with $\mathcal{O}_{-1} = F + \frac{1}{1-d}\sqrt{\frac{1+e+a(1-d)(b-ce)}{g(-1+ac(1-d))}}$, called the positive rank-1 preimage, which occurs when the first iterate is equal to F.

$b > b^F$. Increasing parameters e and d might prevent divergent dynamics, while increasing parameters a and b might cause divergent dynamics.

Figure 1 summarizes our analytical results. The left (right) panel presents a two-dimensional bifurcation diagram in the parameter plane (c, b) for $d = 0$ ($d = 0.25$). The cyan area visualises parameter combinations for which the model's dynamics converge to its fundamental steady state. The purple area visualises parameter combinations for which the model's dynamics converge to a non-fundamental steady state. The pink area visualises parameter combinations that generate endogenous dynamics in either the bull or the bear market. The burgundy area visualises parameter combinations that generate endogenous dynamics, switching between bull and bear markets. The dark-burgundy area visualises parameter combinations for which the model's dynamics are divergent. The brown, orange and green-colored areas visualise parameter combinations that generate a period-two, -four, and -eight cycle, respectively.

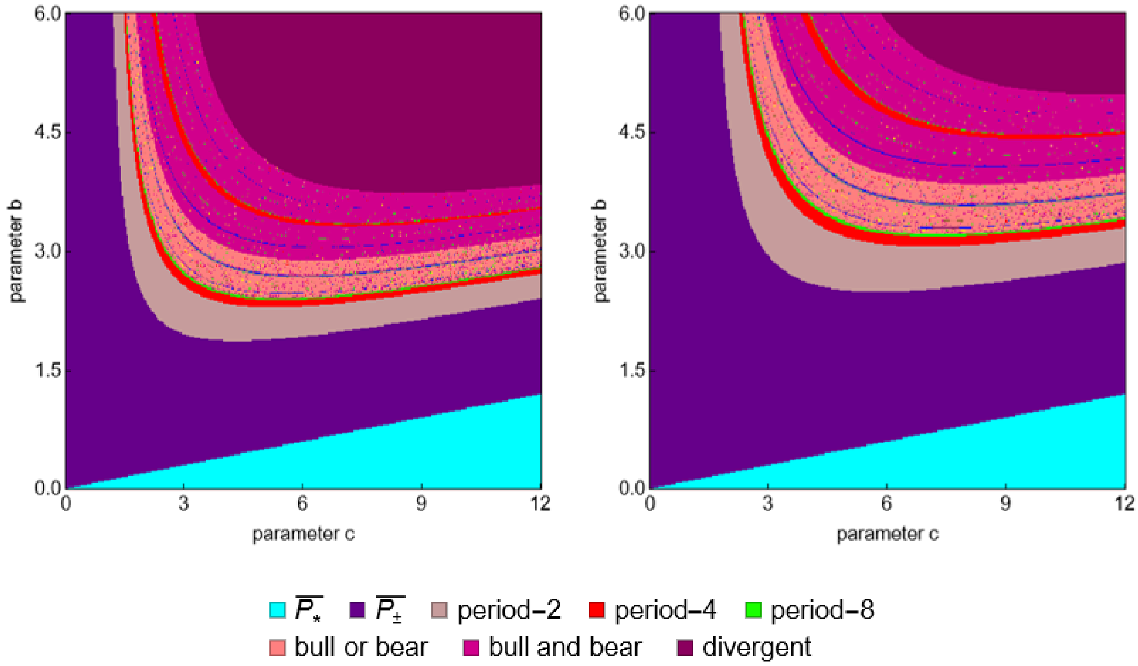

Figure 1: The left (right) panel presents a two-dimensional bifurcation diagram in the parameter plane (c, b) for $d = 0$ ($d = 0.25$).

3.3 Numerical results

Table 1 presents our parameter setting. The parameter values have been selected in order to best visualize the results.

Parameter	Economic significance
$a = 1$	Price adjustment parameter
$b = 6$	Reaction parameter of chartists
$c = 2$	Reaction parameter of fundamentalists
$e = 0.1$	Parameter determining the maximum market share of chartists
$g = 1$	Switching parameter
$F = 100$	Long-run fundamental value

Table 1: Parameter setting.

The left (right) panel of Figure 2 shows the bifurcation of the price versus parameter b for $d = 0$ ($d = 0.25$). The price converges to its fundamental steady state $\overline{P_\star} = F = 100$, represented in cyan, as long as the stability conditions for the fundamental steady state hold. At the pitchfork bifurcation $b^P = 0.2$, the fundamental steady state becomes unstable and two locally stable non-fundamental steady states emerge, shown in purple. For $d = 0$, $P_\pm = F \pm \sqrt{\frac{b-0.2}{2}}$ and for $d = 0.25$, $P_\pm = F \pm \frac{4}{3}\sqrt{\frac{b-0.2}{2}}$. A period-doubling bifurcation occurs at $b^{PD2} = 2.4$ for $d = 0$ and at $b^{PD2} = 4.6$ for $d = 0.25$. The two non-fundamental steady states then become unstable, giving rise to a locally stable period-two cycle, represented in brown. This period-two cycle becomes unstable at $b^{PD4} = 3.5$ for $d = 0$ and $b^{PD4} \approx 7.77$ for $d = 0.25$, giving rise to a locally stable period-four cycle, shown in orange. By further increasing the value of b, a sequence of period doubling occurs, leading to chaotic dynamics, represented in pink. A homoclinic bifurcation of the fundamental steady state occurs at $b^H = 5.04$ for $d = 0$ and $b^H \approx 11.76$ for $d = 0.25$. For $b^P < b < b^H$, two symmetric attractors coexist. Depending on the initial value, the system is either attracted to the attractor above the fundamental value, i.e., in the bull market, depicted in a darker shade, or to the attractor below the fundamental value, i.e., in the bear market, depicted in a lighter shade. The dark areas visualise initial values leading to convergence to the attractor in the bull market, while the light areas visualise initial values leading to convergence to the attractor in the bear market. At b^H, the distinct bear and bull markets merge, and for $b > b^H$ we have dynamics in the bull and bear markets, shown in burgundy.[3] Since $ac\delta \geq 2$, we have no divergent dynamics here.[4]

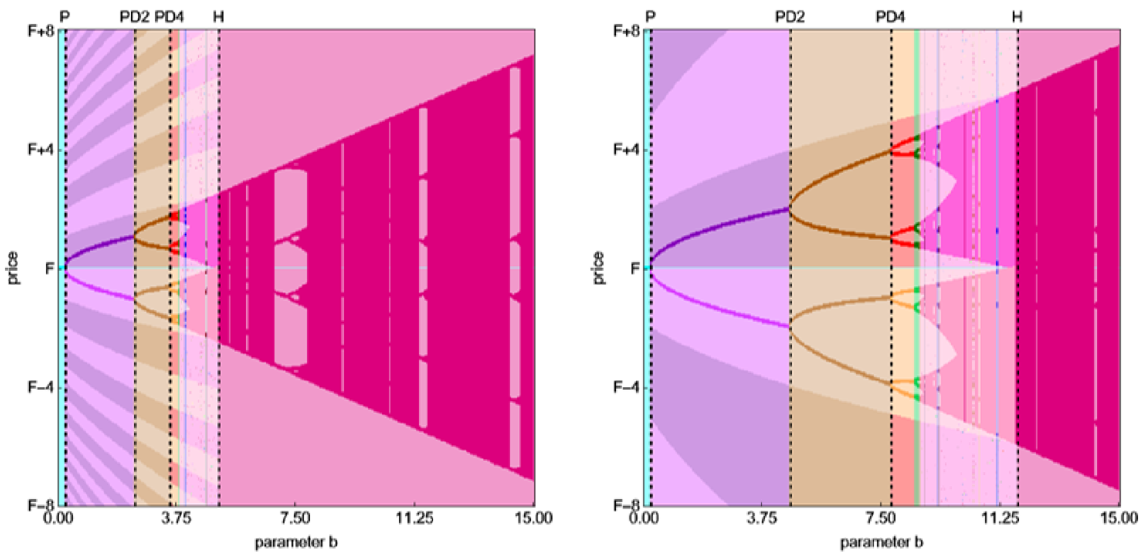

Figure 2: The left (right) panel shows the bifurcation diagram in which the price is depicted versus parameter b for $d = 0$ ($d = 0.25$).

The left (right) panel of Figure 3 shows the time evolution of the price and the perceived fundamental value (in gray) for $d = 0$ ($d = 0.25$). The top, middle, and bottom panels are based on our parameter setting, except that $b = 6$, $b = 10$, and $b = 15$, respectively. We can see from the left panel of Figure 2 that for the case $d = 0$, a homoclinic bifurcation of the fundamental steady state occurs at $b = 5.04$, generating endogenous dynamics switching between bull and bear markets. These dynamics are depicted in the

[3]Note that coexisting period n-cycles are still possible after this point.

[4]Note that if $d = 0$ (for this parameter setting), the unstable period two cycle exists at infinity.

left panels of Figure 3. For the case $d = 0.25$, however, as shown in the right panel of Figure 2 at $b = 6$, we have a period-two cycle either in the bull or in the bear market, visible in the top right panel of Figure 3. For $b = 10$, the right panel of Figure 2 shows that for the case $d = 0.25$, we have endogenous dynamics either in the bull or in the bear market, depicted in the middle right panel of Figure 3. For $b = 15$, the right panel of Figure 2 shows that we have endogenous dynamics switching between the bear and the bull market, since for $d = 0.25$, a homoclinic bifurcation of the fundamental steady state has occurred at $b \approx 11.76$, presented in the bottom right panel of Figure 3. For $b = 15$, the average time span to the price cross F is equal to 2 time steps for $d = 0$, while for $d = 0.25$ it is equal to 7 time steps.

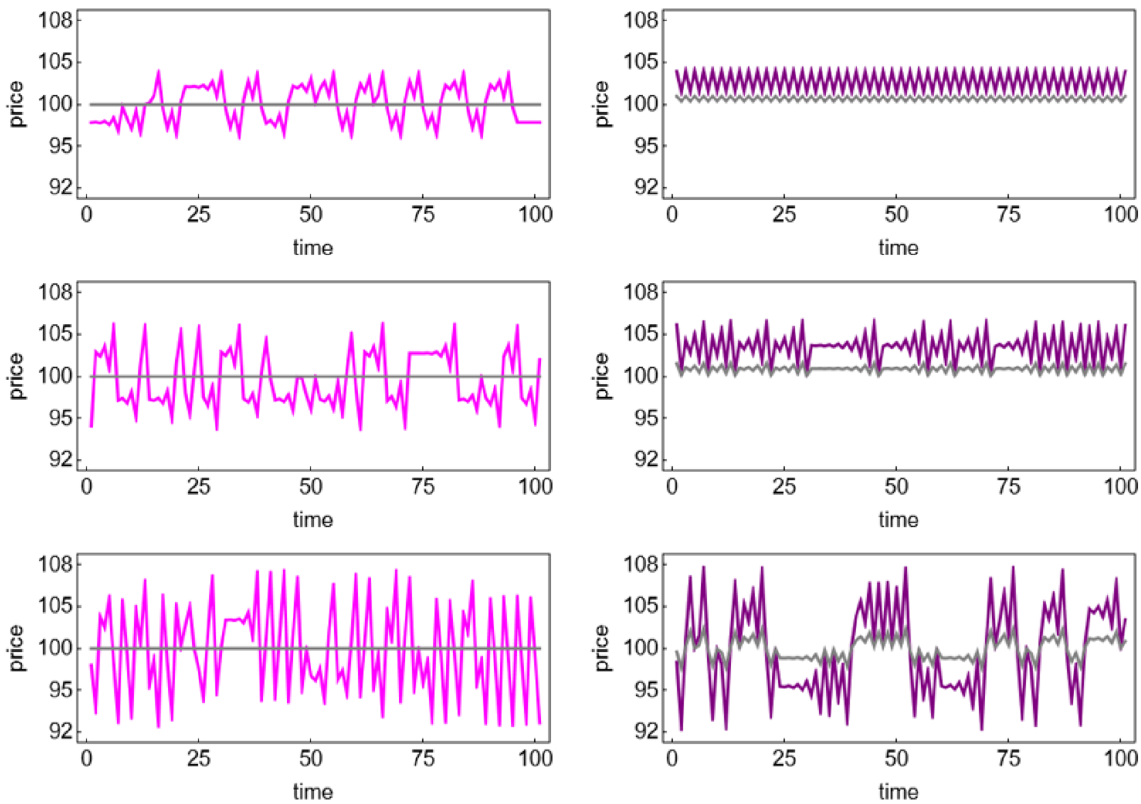

Figure 3: The left panels show the evolution of the price (pink) and the perceived fundamental (gray) for $d = 0$. The right panels show the evolution of the price (purple) and the perceived fundamental (gray) for $d = 0.25$. The top, middle, and bottom panels are based on our parameter setting, except that $b = 6$, $b = 10$, and $b = 15$, respectively.

Let us further illustrate the implication of the homoclinic bifurcation of the fundamental steady state. Figure 4 shows the evolution of the price in the phase diagram for $d = 0.25$ and different values of $b = 4, 6, 10, 15$. The green dots indicate the steady states. For $b = 4$, the price dynamics converge to the non-fundamental steady state. For $b = 6$, the price dynamics converge to a stable period-two cycle denoted by the two orange dots. For $b = 10$, we observe endogenous price dynamics, albeit trapped in the bull market. In fact, before the homoclinic bifurcation of the fundamental steady state, depending on the initial value, the price dynamics fluctuate either between the local maximum value MAX and its image $f(MAX)$ or between the local minimum value MIN and its image $f(MIN)$. This means that the price dynamics are either trapped in a bull market $J_{MAX} = [f(MAX), MAX]$ or in a bear market $J_{MIN} = [MIN, f(MIN)]$. At the homoclinic bifurcation of the fundamental steady state, MIN and MAX are mapped into the unstable fundamental steady state F, i.e., when the third iteration is equal to F, $f(MIN) = F = f(MAX)$. After the homoclinic bifurcation of the fundamental steady state, the price dynamics fluctuate between $[MIN, MAX]$, i.e., the price dynamics switch between bull and bear markets, visible for the case $b = 15$.

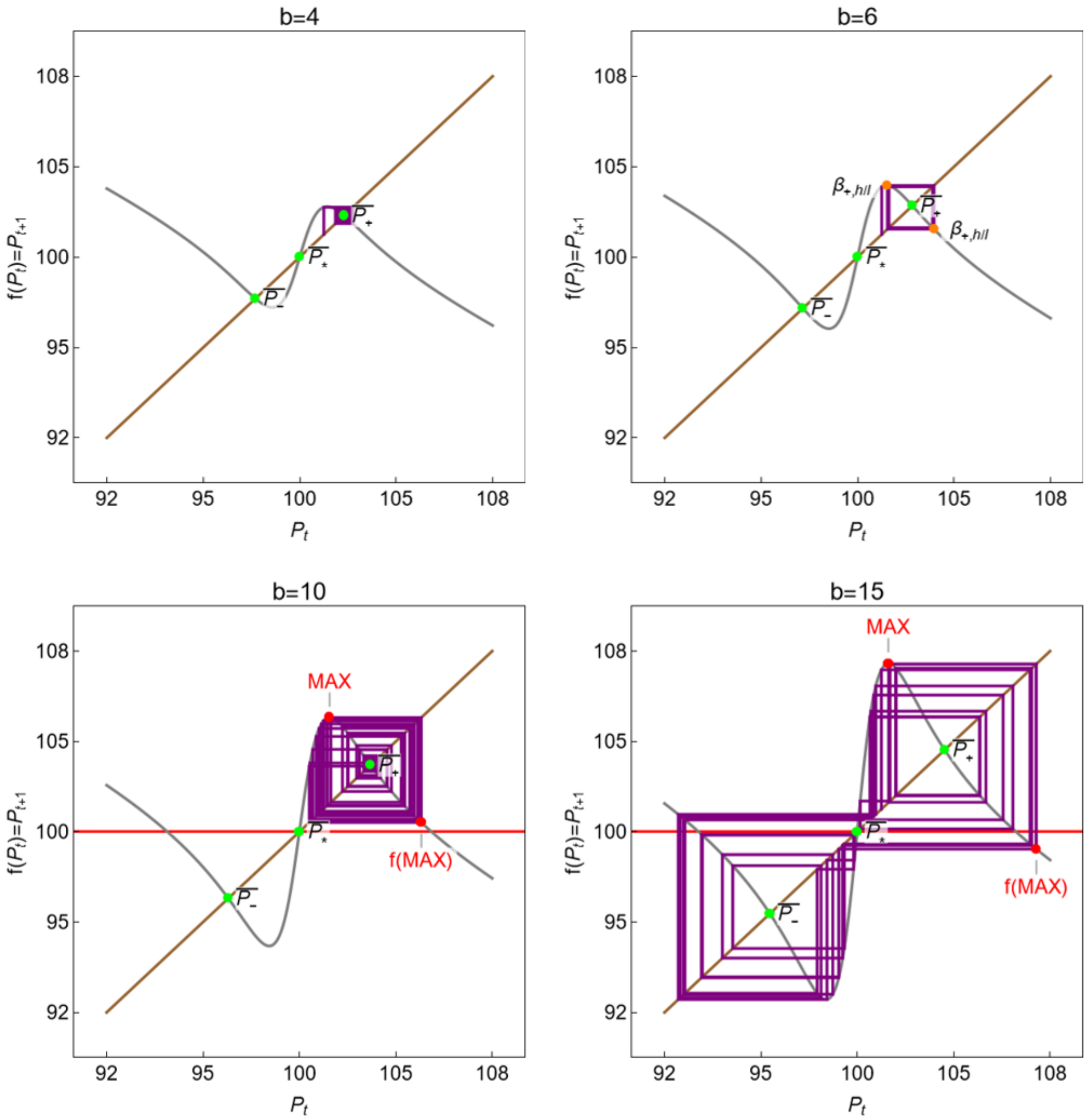

Figure 4: The panels present the evolution of the price in the phase diagram for $d = 0.25$ and different values of $b = 4, 6, 10, 15$. The green dots indicate the steady states. The orange dots indicate the periodic point of the period-two cycle.

4 Moving averages

4.1 The model with moving averages

We now assume that agents take into account price information over the period $[t - k, t]$ to form their expectation of the current fundamental value. More precisely, they believe that the current fundamental value reacts to the deviation between the moving average of length $k+1$, defined as $\frac{1}{k+1}\sum_{i=0}^{k} P_{t-i}$, from F.[5] Accordingly,

$$F_t = F + d\left(\frac{1}{k+1}\sum_{i=0}^{k} P_{t-i} - F\right) = (1-d)F + d\frac{1}{k+1}\sum_{i=0}^{k} P_{t-i} \tag{15}$$

4.2 Analytical results

Combining (1) to (6), and (15) reveals that our model is now driven by a $(k+1)$-th order nonlinear deterministic difference equation.

$$P_{t+1} = P_t + a(F_t - P_t)\frac{cg(F_t-P_t)^2 + (ce-b)}{1+e+g(F_t-P_t)^2}, \tag{16}$$

with $F_t = (1-d)F + d\frac{1}{k+1}\sum_{i=0}^{k} P_{t-i}$.[6] Setting $P_{t+1} = P_t = P_{t-1} = ... = P_{t-k} = \overline{P}$, we find that map (16) possesses three steady states: A fundamental steady state given by

$$FSS_\star = (\overline{P_{0,\star}}, \overline{P_{1,\star}}, ..., \overline{P_{-k,\star}}) = (F, F, ..., F) \tag{17}$$

and two non-fundamental steady states given by

$$NFSS_\pm = (\overline{P_{0,\pm}}, \overline{P_{-1,\pm}}, ..., \overline{P_{-k,\pm}}) = \left(F \pm \frac{1}{1-d}\sqrt{\frac{b-ce}{cg}}, \overline{P_{0,\pm}}, ..., \overline{P_{0,\pm}}\right). \tag{18}$$

The fundamental steady state always exist whereas the two non-fundamental steady states only exist for $b \geq ce$.

Let us next study the local stability properties of the steady states. The Jacobian matrix of map (16), evaluated at $FSS_\star$, yields

$$J(FSS_\star) = \begin{bmatrix} 1 + a\left(1-\frac{d}{k+1}\right)\frac{b-ce}{1+e} & -\frac{ad(b-ce)}{(k+1)(1+e)} & \dots & -\frac{ad(b-ce)}{(k+1)(1+e)} & -\frac{ad(b-ce)}{(k+1)(1+e)} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \tag{19}$$

from which we get the characteristic polynomial

$$P(\lambda) = \lambda^{k+1} - \left(1 + a\left(1-\frac{d}{k+1}\right)\frac{b-ce}{1+e}\right)\lambda^k + \frac{ad(b-ce)}{(k+1)(1+e)}(\lambda^{k-1} + ... + 1). \tag{20}$$

Two necessary local stability conditions require that:

$$P(1) = \frac{a(1-d)(ce-b)}{1+e} > 0 \tag{21}$$

$$(-1)^{k+1}P(-1) = \begin{cases} 2 + \left(1-\frac{d}{k+1}\right)a\frac{(b-ce)}{1+e} > 0 & \text{if k is even} \\ 2 + a\frac{b-ce}{1+e} > 0 & \text{if k is odd} \end{cases}. \tag{22}$$

[5]See Chiarella and He (2002) and Chiarella et al. (2006a,b)for economic applications of the moving average.

[6]To transform the $(k+1)$-th order nonlinear deterministic difference equation, which depends on the past $k+1$ prices $P_t, P_{t-1}, ..., P_{t-k}$, into a map, we introduce k auxiliary variables corresponding to $P_{t-1}, P_{t-2}, ..., P_{t-k}$. For clarity, we use these past prices directly as the auxiliary variables.

The Jacobian matrix of map (29), evaluated at $NFSS_\pm$, yields

$$J(NFSS_\pm) = \begin{bmatrix} 1-2ac\left(1-\frac{d}{k+1}\right)\frac{b-ce}{b+c} & \frac{2acd(b-ce)}{(k+1)(b+c)} & \dots & & \frac{2acd(b-ce)}{(k+1)(b+c)} & \frac{2acd(b-ce)}{(k+1)(b+c)} \\ 1 & 0 & \dots & 0 & 0 & \\ 0 & 1 & \ddots & 0 & 0 & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \dots & 1 & 0 & \end{bmatrix} \tag{23}$$

from which we get the characteristic polynomial

$$P(\lambda) = \lambda^{k+1} - \left(1-2ac\left(1-\frac{d}{k+1}\right)\frac{b-ce}{b+c}\right)\lambda^k - \frac{2acd(b-ce)}{(k+1)(b+c)}(\lambda^{k-2}+...+1). \tag{24}$$

Two necessary local stability conditions require that:

$$P(1) = \frac{2ac(1-d)(b-ce)}{b+c} > 0 \tag{25}$$

$$(-1)^{k+1}P(-1) = \begin{cases} 2+2(1-\frac{d}{k+1})\frac{ac(ce-b)}{b+c} > 0 & \text{if k is even} \\ 2+2\frac{ac(ce-b)}{b+c} > 0 & \text{if k is odd} \end{cases}. \tag{26}$$

Proposition 3 *Map (16) may possess up to three steady states, namely* $FSS_\star = (\overline{P_{0,\star}}, \overline{P_{-1,\star}}..., \overline{P_{-k,\star}}) = (F, F, ..., F)$ *and* $NFSS_\pm = (\overline{P_{0,\pm}}, \overline{P_{-1,\pm}}, ..., \overline{P_{-k,\pm}}) = \left(F \pm \frac{1}{1-d}\sqrt{\frac{b-ce}{cg}}, \overline{P_{0,\pm}}, ..., \overline{P_{0,\pm}}\right)$. *$FSS_\star$ always exists. A necessary condition for $FSS_\star$ to be locally asymptotically stable is that $ce - \frac{2(1+e)}{a(1-\frac{d}{k+1})} < b < ce$ for even k and $ce - \frac{2(1+e)}{a} < b < ce$ for odd k. $NFSS_\pm$ exist if $b > ce$ and lie symmetrically around $FSS_\star$. A necessary condition for $NFSS_\pm$ to be locally asymptotically stable is that $ce < b < ce + \frac{c(1+e)}{-1+ac(1-\frac{d}{k+1})}$ or $ce < b$ and $ac(1-\frac{d}{k+1}) \leq 1$ for even k and $ce < b < ce + \frac{c(1+e)}{-1+ac}$ or $ce < b$ and $ac \leq 1$ for odd k.*

Let us discuss the main economic implications of Proposition 3. Given the difficulty of analysing a k-dimensional characteristic polynomial, Proposition 3 can at least provide necessary conditions for the steady state to be locally stable.

- **Fundamental steady state**. A necessary stability condition requires that $b < ce$. Since the non-fundamental steady states emerge at $b = ce$, this may suggest that a pitchfork bifurcation may occur at that point.
- **Non-fundamental steady state**. Non-fundamental steady state prices are the same as those encountered in the previous section, and in particularly are independent of k. Depending on whether k is even or odd, this necessary stability condition hinges on d. For even k, this necessary condition hinges on d, requiring that $b < ce + \frac{c(1+e)}{-1+ac(1-\frac{d}{k+1})}$, decreasing with k. For odd k, this necessary condition does not hinge on d and is equivalent to the case $d = 0$, requiring that $b < ce + \frac{c(1+e)}{-1+ac}$.

4.3 Numerical results

We use the same parameter setting as before, presented in Table 1. Figure 5 shows the bifurcation diagram plotting price versus parameter b for $d = 0$ (magenta) and $d = 0.6$ (purple) for different values of $k = 0, 1, 2, 3, 10, 15$.

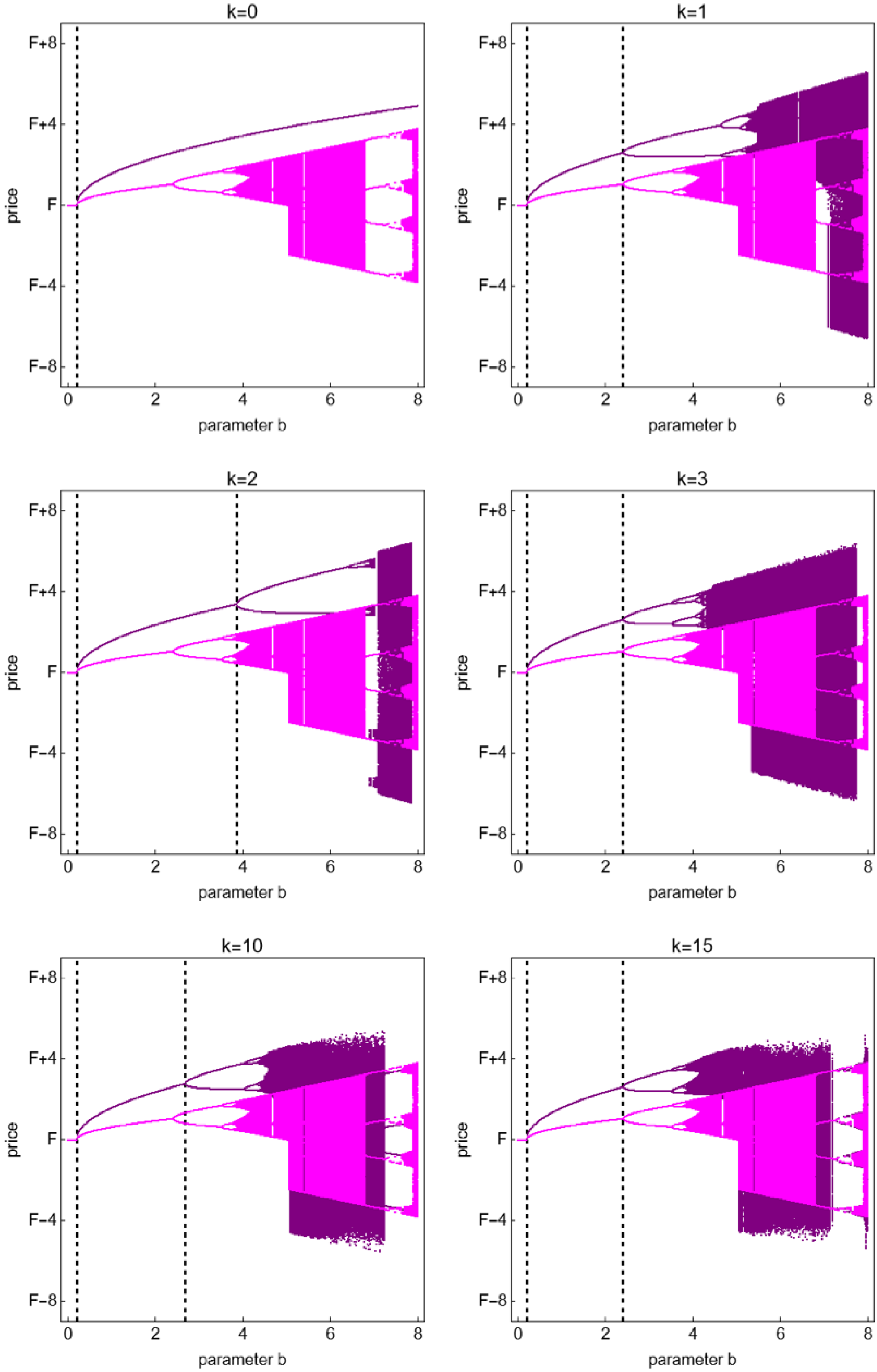

Figure 5: The panels show the bifurcation diagrams in which the price is depicted versus parameter b for $d = 0$ (magenta) and $d = 0.6$ (purple) for different values of $k = 0, 1, 2, 3, 10, 15$.

The bifurcation diagrams were generated using a combination of initial values $(P_0, P_1, \ldots, P_k) = (\overline{P_\star}, \overline{P_\star}, \ldots, \overline{P_\star}+0.01)$, resulting in dynamics that occur only in the bull market before the homoclinic bifurcation of the fundamental steady state. For the alternative initial values $(P_0, P_1, \ldots, P_k) = (\overline{P_\star}, \overline{P_\star}, \ldots, \overline{P_\star} - 0.01)$, a mirror version of the bifurcation diagram would be obtained, i.e., resulting in dynamics that occur only in the bear market before the homoclinic bifurcation of the fundamental steady state. All bifurcation diagrams have in common that the price dynamics converge to its fundamental steady state if $b < \tilde{b} = 0.2$. At the bifurcation point $b = 0.2$, the fundamental steady state becomes unstable and two locally stable non-fundamental steady states emerge. For $d = 0.6$ and $k = 0$, the two non-fundamental steady states are locally stable if they exist since $ac(1 - d) = 0.8 < 1$. For $d = 0.6$ and odd ks, $k = 2$, $k = 10$ a period-doubling bifurcation occurs at $b = 2.4$, $b \approx 3.87$ and $b \approx 2.67$, respectively. The two non-fundamental steady states then become unstable, giving rise to a locally stable period-two cycle. Increasing the value of b leads to a sequence of period-doubling bifurcations, causing endogenous price dynamics initially in the bull market, and later in both the bull and bear markets. For $d = 0.6$ at lower values of ks, the homoclinic bifurcation of the fundamental steady state occurs for higher values of parameter b. Note that for $d = 0.6$, the price dynamics fluctuate in a higher price range compared to $d = 0$.

Figure 6 shows the evolution of the price (purple) and the perceived fundamental value (gray) for $d = 0.6$ and different values of $k = 0, 1, 2, 3, 10, 15$.

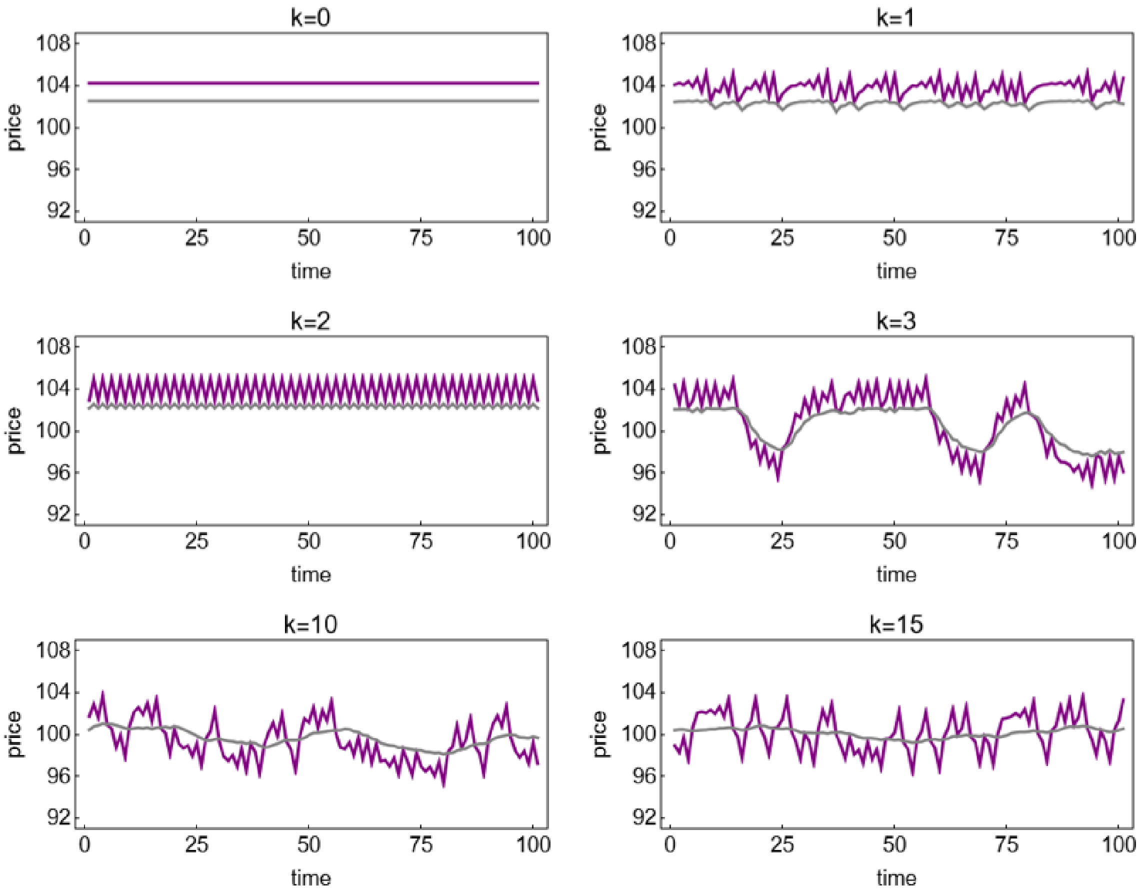

Figure 6: The panels show the evolution of the price (purple) and the perceived fundamental value (gray) for $d = 0.6$ and different values of $k = 0, 1, 2, 3, 10, 15$.

For $k = 0$, the dynamics remain at the non-fundamental steady state. For $k = 1$, we have endogenous dynamics in the bull market. For $k = 2$, we observe a period-two cycle in the bull market. For $k = 3$, we have endogenous dynamics, switching between the bear and the bull market. The time span before the price returns to its long-run fundamental value is 16 time steps on average. In the middle right panel, we observe that the price fluctuates for a while in the bull market before abruptly switching to the bear market. It quickly recovers and fluctuates again in the bull market before falling rapidly again. After a brief peak, the price fluctuates in the bear market. This indicates us that prices show periods of fluctuation in the bull and bear markets, but that abrupt and rapid shifts are possible. For $k = 10$ and $k = 15$, the perceived fundamental value flattens out and deviates less strongly from its long-run fundamental value. The average time for the price to return to its long-run fundamental value is 6 and 5 time steps, respectively.

5 Exponential moving average

5.1 The model with exponential moving average

We now assume that agents believe that the current fundamental value reacts to the deviation of the average price from F.

$$F_t = F + d(X_t - F). \tag{27}$$

The average price is now modelled as an exponential moving average.[7] Accordingly,

$$X_t = memP_t + (1 - mem)X_{t-1} \tag{28}$$

is a weighted average of the past average price and the current price, where $0 < mem < 1$ is a memory parameter. The lower mem, the higher the weight given to the past average price. For $mem \to 1$, we have $X_t \to P_t$ and for $mem \to 0$, we have $X_t \to X_{t-1}$.[8] Note that the case $mem \to 0$ refers to the case where $d = 0$ and $mem \to 1$ refers to the case $d > 0$ in our first core setup.

5.2 Analytical results

Combining (1) to (6), (27), and (28) reveals that our model is now driven by a two-dimensional nonlinear deterministic map

$$M := \begin{cases} P_{t+1} &= P_t + a(F_t - P_t)\frac{cg(F_t-P_t)^2+(ce-b)}{1+e+g(F_t-P_t)^2} \\ X_{t+1} &= memP_{t+1} + (1 - mem)X_t \end{cases}, \tag{29}$$

with $F_t = F + d(X_t - F)$.

Setting $P_{t+1} = P_t = \overline{P}$ and $X_{t+1} = X_t = \overline{X}$, we find out that map (29) possesses three steady states: A fundamental steady state given by

$$FSS_\star = (\overline{P_\star}, \overline{X_\star}) = (F, F) \tag{30}$$

and two non-fundamental steady states

$$NFSS_\pm = (\overline{P_\pm}, \overline{X_\pm}) = \left(F \pm \frac{1}{1-d}\sqrt{\frac{b-ce}{cg}}, F \pm \frac{1}{1-d}\sqrt{\frac{b-ce}{cg}}\right). \tag{31}$$

[7]Note: This updating rule of the average price is obtained by the limit case of the geometry decay process where the memory decay rate tends to infinity.

[8]See Hommes et al. (2012) for economic applications of the exponential moving average.

The fundamental steady state always exist whereas the two non-fundamental steady states only exist for $b \geq ce$.

Let us now study the local stability properties of the steady states.

The Jacobian matrix of map (29) evaluated at $FSS_\star$ yields

$$J(FSS_\star) = \begin{bmatrix} 1 + a\frac{b-ce}{1+e} & -\frac{ad(b-ce)}{1+e} \\ mem\left(1 + a\frac{b-ce}{1+e}\right) & 1 - mem - mem\frac{ad(b-ce)}{1+e}) \end{bmatrix} \tag{32}$$

from which we get the characteristic polynomial

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{33}$$

where $a_1 = -\left(1 + (1 - mem) + a(1 - dmem)\frac{b-ce}{1+e}\right)$ and $a_2 = (1 - mem)\left(1 + \frac{a(b-ce)}{1+e}\right)$. Necessary and sufficient conditions ensuring that the two eigenvalues of (33) are less than one in modulus are given by $(i) 1 + a_1 + a_2 > 0$, $(ii) 1 - a_1 + a_2 > 0$ and $(iii) 1 - a_2 > 0$. Condition (i) requires that

$$\frac{a(1-d)mem(ce-b)}{1+e} > 0. \tag{34}$$

Condition (ii) requires that

$$2(1 + (1 - mem)) + \frac{a(b-ce)(1 + (1 - mem) - dmem)}{1+e} > 0. \tag{35}$$

Solving for b results in

$$ce - \frac{2(1+e)}{a(1 - \frac{mem}{1+(1-mem)}d)} < b < ce. \tag{36}$$

Condition (iii) $mem + \frac{a(ce-b)(1-mem)}{1+e} > 0$ always holds if $b < ce$.

The Jacobian matrix of map (29) evaluated at $NFSS_\pm$ yields

$$J(NFSS_\pm) = \begin{bmatrix} 1 - 2ac\frac{b-ce}{b+c} & \frac{2acd(b-ce)}{b+c} \\ mem\left(1 - 2ac\frac{b-ce}{b+c}\right) & 1 - mem + mem\left(\frac{2acd(b-ce)}{b+c}\right) \end{bmatrix} \tag{37}$$

from which we get the characteristic polynomial

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2, \tag{38}$$

where $a_1 = -\left(1 + (1 - mem) - 2ac(1 - dmem)\frac{b-ce}{b+c}\right)$ and $a_2 = (1-mem)\left(1 - \frac{2ac(b-ce)}{b+c}\right)$. Condition (i) requires that

$$\frac{2ac(1-d)mem(b-ce)}{b+c} > 0. \tag{39}$$

Condition (ii) requires that

$$2 + 2(1 - mem) + 2\frac{ac(ce-b)(1 + (1 - mem) - dmem)}{b+c} > 0. \tag{40}$$

Solving for b results in

$$ce < b < ce + \frac{c(1+e)}{-1 + ac(1 - \frac{mem}{1+(1-mem)}d)} \quad \text{for} \quad ac(1 - \frac{mem}{1+(1-mem)}d) > 1, \quad ce < b \quad \text{otherwise} \tag{41}$$

Condition (iii) $mem + \frac{2ac(b-ce)(1-mem)}{b+c} > 0$ always holds if $b > ce$.

Proposition 4 *Map (8) may possess up to three steady states: A fundamental steady state* $FSS_\star = (\overline{P_\star}, \overline{X_\star}) = (F, F)$ *and two non-fundamental steady states* $NFSS_\pm = (\overline{P_\pm}, \overline{X_\pm}) = \left(F \pm \frac{1}{1-d}\sqrt{\frac{b-ce}{cg}}, \overline{P_\pm}\right)$. *The fundamental steady state always exists. For* $b^{PD1} < b < b^P$, $FSS_\star$ *is locally stable. A period-doubling bifurcation occurs at* $b^{PD1} = ce - \frac{2(1+e)}{a(1-\frac{mem}{1+(1-mem)}d)}$. *At* $b^P = ce$, *a pitchfork bifurcation occurs and two non-fundamental steady states emerge and lie symmetrically around* $\overline{P_\star}$. $NFSS_\pm$ *are locally stable if either* $b^P < b < b^{PD2}$ *or* $b^P < b$ *and* $ac(1-\frac{mem}{1+(1-mem)}d) \leq 1$. *A period-doubling bifurcation occurs at* $b^{PD2} = ce + \frac{c(1+e)}{-1+ac(1-\frac{mem}{1+(1-mem)}d)}$ *if* $ac(1-\frac{mem}{1+(1-mem)}d) > 1$.

Let us discuss the main economic implications of Proposition 4.

- **Fundamental steady state**. As in Section 3, the fundamental steady state can lose its local stability via a period-doubling bifurcation or via a pitchfork bifurcation. The occurrence of a period-doubling bifurcation requires extreme values of c. At $b = ce$ a pitchfork bifurcation occurs.

- **Non-fundamental steady state**. Non-fundamental steady state prices are equal to those encountered in the previous section, and in particular are independent of mem. Increasing parameter mem can be beneficial for their local stability. Note that $ce + \frac{c(1+e)}{-1+ac} < b^{PD2} < ce + \frac{c(1+e)}{-1+ac(1-d)}$, i.e., lying between the two extreme cases $mem \to 0$ and $mem \to 1$.

5.3 Numerical results

We use the same parameter setting as before, presented in Table 1. The left (right) panel of Figure 7 shows the bifurcation of price versus parameter b for $d = 0$, depicted in magenta, and $d = 0.6$, depicted in purple, for $mem = 0.15$ ($mem = 0.05$). The bifurcation diagrams were generated using the initial values $(\overline{P_0}, \overline{X_0}) = (\overline{P_\star} + 0.01, \overline{X_\star})$. For the alternative initial conditions $(\overline{P_0}, \overline{X_0}) = (\overline{P_\star} - 0.01, \overline{X_\star})$, a mirrored version of the bifurcation diagram would be obtained. The price dynamics converge to the fundamental steady state if $b < b^P = 0.2$. At the bifurcation point $b = 0.2$, the fundamental steady state becomes unstable and two locally stable non-fundamental steady states emerge. For $mem = 0.15$, $mem = 0.05$ a period-doubling bifurcation occurs at $b^{PD2} \approx 2.64$, $b^{PD2} \approx 2.47$, respectively. By increasing b, a sequence of period doubling bifurcations occurs, leading to endogenous price dynamics initially only in the bull market and later in both the bear and bull markets. It is noteworthy that for lower mem, the maximum value of the price after the homoclinic bifurcation of the fundamental steady state is lower than endogenous dynamics in the bull market. The reason is that if prices fluctuate permanently above F, the perceived fundamental value fluctuates at a high level, driving prices to higher values. After the homoclinic bifurcation of the fundamental steady state, if mem is sufficiently low, due to a high memory of the average price, the perceived fundamental value is really close to F, preventing prices from reaching high values.[9]

[9]Note that if we increased k further in the model with moving averages we would observe the same phenomenon.

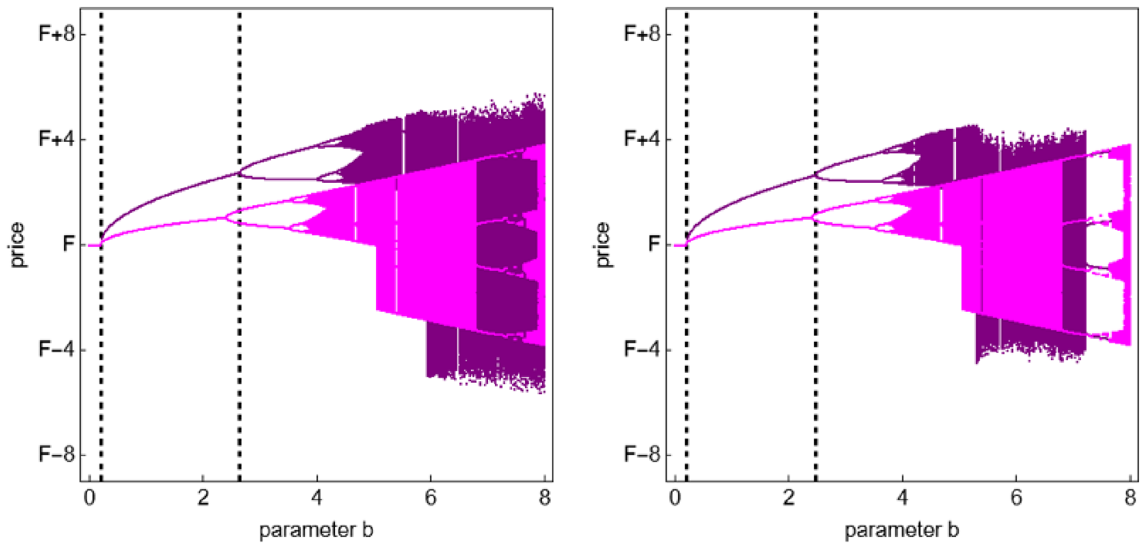

Figure 7: The left panel shows the bifurcation diagram plotting price versus parameter b for $d = 0$ (magenta) and $d = 0.6$ (purple) for $mem = 0.15$. The right panel shows the bifurcation diagram plotting price versus parameter b for $d = 0$ (magenta) and $d = 0.6$ (purple) for $mem = 0.05$.

The left (right) panel of Figure 8 shows the evolution of the price (purple) and the perceived fundamental value (gray) for $d = 0.6$ and $mem = 0.15$ ($mem = 0.05$). For $mem = 0.15$, the average time span to the price cross F is 13 time steps. In the left panel, we observe that the price fluctuates for a while in the bull market before gradually shifting to the bear market, where the price then fluctuates for a while before gradually rising again. This suggests that the changes between bull and bear markets are not abrupt. For $mem = 0.05$, the perceived fundamental value flattens out and deviates less strongly from its long-run fundamental value. The average time span to the price cross F is 6 time steps.

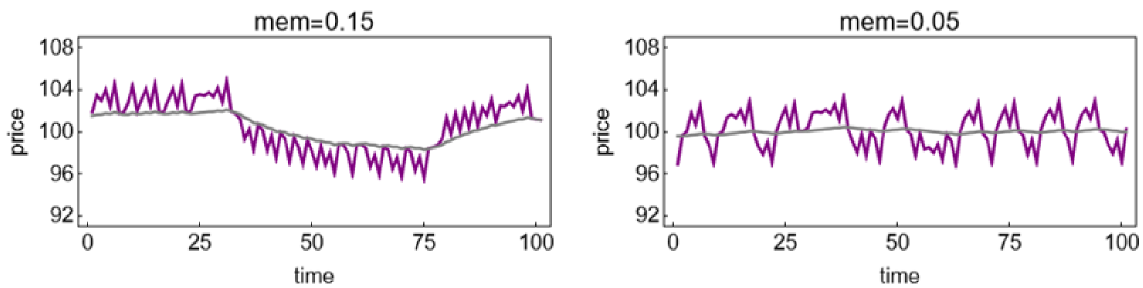

Figure 8: The left panel shows the evolution of the price (purple) and the perceived fundamental value (gray) for $d = 0.6$ and $mem = 0.15$. The left panel shows the evolution of the price (purple) and the perceived fundamental value (gray) for $d = 0.6$ and $mem = 0.05$.

6 Conclusion

The starting point of our paper was that agents do not know the true current fundamental value. As a workhorse, we used a simple nonlinear model in which speculators either opt for a technical or a fundamental trading rule, depending on market conditions. Following the anchoring and adjustment heuristic of Tversky and Kahneman (1974), we assumed that agents believe that the current fundamental value fluctuates around the known long-run fundamental value, serving as an anchor. To adjust their perception, they use a weighted average of the current price and the long-run fundamental value. It turns out that a one-dimensional nonlinear map, possessing one fundamental steady state and two non-fundamental steady states, governs the dynamics of our model. We proved analytically that an increase in the reaction strength of technical traders can lead to a sequence of bifurcations, which can cause the price to fluctuate in bull and bear markets or even diverge. However, if agents have a heightened belief that the current fundamental value deviates more strongly from its long-run fundamental value, this could cause the price dynamics to be trapped above or below the fundamental steady states. Numerically, we showed that when the perception of the fundamental value is not constant, the price fluctuates in a higher price range and the time span of a bubble is longer.

Then, in a second and third setup, we proposed that the fundamental value perception is a weighted average between the known long-run fundamental value and an average price of the k past prices and the exponential moving average. It turned out that price and perceived fundamental value fluctuate less statically around the long-run fundamental value.

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Appendix A: Proof of Proposition 2

In this appendix, we prove the secondary bifurcations of our core model. Specifically, we compute the points of the model's period-two cycle, their local stability condition, the bifurcation point of the homoclinic bifurcation of the fundamental steady state, and the bifurcation point of the final bifurcation.

Second iterate

By defining $y_t := P_t - F$, $\delta = 1 - d$ and rearranging the terms, we can reformulate map (8) as follows

$$y_{t+1} = y_t \frac{Ay_t^2 + B}{1 + e + g\delta^2 y_t^2}, \tag{42}$$

with $A = (1 - ac\delta)g\delta^2$ and $B = 1 + e + a(b - ce)\delta$.

The second iterate of this map is then given by

$$y_{t+2} = y_t \frac{Ay_t^2 + B}{1 + e + g\delta^2 y_t^2} \frac{B + A\frac{y_t^2(B+Ay_t^2)^2}{(1+e+g\delta^2 y_t^2)^2}}{1 + e + g\delta^2 \frac{y_t^2(B+Ay_t^2)^2}{(1+e+g\delta^2 y_t^2)^2}} . \tag{43}$$

Period-two cycles

Setting $y_{t+2} = y_t = \overline{y}$, we obtain

$$\overline{y}((A - g\delta^2)\overline{y}^2 + B - D)((A + g\delta^2)\overline{y}^2 + B + D)(A^2\overline{y}^4 + C\overline{y}^2 + D^2) = 0, \tag{44}$$

with $C = AB + (1 + e)g\delta^2$ and $D = 1 + e$.

By solving the first two terms with respect to $\overline{y}$, we obtain the steady states $\overline{y_\star}$ and $\overline{y_\pm}$. By solving the third term with respect to $\overline{y}$, we obtain the periodic points of the period-two cycle, unstable if $b > b^{PD1}$ [10]

$$\alpha_\pm = \pm\sqrt{-\frac{B + D}{A + g\delta^2}} . \tag{45}$$

[10] Note that this period-two cycle only exists if $ac\delta > 2$ (if $b > b^{PD1}$)

By solving the fourth term with respect to $\overline{y}$, we obtain the periodic points of the period-two cycles, existing if $b > b^{PD2}$ and are locally stable if $b < PD4$, i.e., [11]

$$\beta_{\pm,h/l} = \pm\frac{1}{\sqrt{2}}\sqrt{\frac{-C \pm \sqrt{C^2 - 4A^2D^2}}{A^2}}. \tag{46}$$

The partial first-order derivative of map (43), evaluated at $\beta_{\pm,h/l}$, reads

$$\frac{dy_{t+2}}{dy_t}\left(\beta_{\pm,h/l}\right) = 1 + 2\frac{(E + c(1 + e - a\delta E))(-4 + e(-4 + 3ac\delta) + a\delta(c(3 + a\delta E) - E))}{(1 + e)(c(1 + e) + E)(-1 + ac\delta)}, \tag{47}$$

with $E = b - ce$.

This period-two cycle is locally stable if $|\frac{dy_{t+2}}{dy_t}\left(\beta_{\pm,h/l}\right)| < 1$, resulting in

$$ce + \frac{c(1 + e)}{-1 + ac\delta} < b < ce + \frac{(1 + e)}{-1 + ac\delta}\left(\frac{5 + \sqrt{5}\sqrt{5 + 4ac\delta(ac\delta - 2)}}{2a\delta} - c\right) \quad . \tag{48}$$

Homoclinic and final bifurcation

To compute the local maximum y_0^{MAX} and minimum y_0^{MIN}, we solve $\frac{\partial y_{t+1}}{\partial y_t} = 0$, resulting in

$$y_0^{MAX,MIN} = \pm\frac{1}{\sqrt{2}}\sqrt{-\frac{3AD - Bg\delta^2 + G}{Ag\delta^2}}, \tag{49}$$

with $G = \sqrt{(AD - Bg\delta^2)(9AD - Bg\delta^2)}$.

The corresponding local maximum and minimum values y_1^{MAX} and y_1^{MIN} are then

$$y_1^{MAX,MIN} = - \pm \frac{1}{\sqrt{2}}\frac{A(3AD + G - 3Bg\delta^2) + \sqrt{-\frac{3AD+G-Bg\delta^2}{Ag\delta^2}}}{g\delta^2(-AD - G + Bg\delta^2)} \quad . \tag{50}$$

To obtain the bifurcation point for the homoclinic bifurcation of the fundamental steady state, the iterate of the local maximum and minimum values should be equal to the unstable fundamental steady state $\overline{y_\star} = 0$, resulting in

$$b^H = ce + \frac{(1 + e)}{-1 + ac\delta}\left(\frac{16 + 13ac\delta(ac\delta - 2) + 2(4 + 3ac\delta(-2 + ac\delta))^{\frac{3}{2}}}{a\delta(5 + 4ac\delta(-2 + ac\delta))} - c\right) \quad . \tag{51}$$

To obtain the value for the final bifurcation, we solve $y_1^{MAX,MIN} = \alpha_\pm$, resulting in:

$$b^F = ce + \frac{2c(1 + e)}{-2 + ac\delta} \quad . \tag{52}$$

[11] Note that this period-two cycle only exists if $b > ce + c\frac{1+e}{-1+ac\delta}$, being the condition that $C^2 > 4A^2D^2$. Note $C < 0$ and $A < 0$.

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